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ON THE SPECTRUM OF THE DISTRIBUTIONAL KERNEL RELATED TO THE RESIDUE

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ABSTRACT. We study the spectrum of the distributional kernel $K_{\alpha,\beta}(x)$, where α and β are complex numbers and x is a point in the space $\mathbb{R}^n$ of the n-dimensional Euclidean space. We found that for any nonzero point ξ that belongs to such a spectrum, there exists the residue of the Fourier transform $(-1)^k\widehat{K_{2k,2k}(\xi)}$, where $\alpha = \beta = 2k$, k is a nonnegative integer and $\xi \in \mathbb{R}^n$.

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1. Introduction. Gel'fand and Shilov [2, pages 253–256] have studied the generalized function P^λ, where

$$P = \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \tag{1.1}$$

is a quadratic form, λ is a complex number, and $p+q = n$ is the dimension of $\mathbb{R}^n$. They found that P^λ has two sets of singularities, namely $\lambda = -1,-2,\dots,-k,\dots$ and $\lambda = -n/2, -n/2-1,\dots,-n/2-k,\dots$, where k is a positive integer. For the singular point $\lambda = -k$, the generalized function P^λ has a simple pole with residue

$$\frac{(-1)^k}{(k-1)!}\delta_1^{(k-1)}(P) \qquad \text{or} \qquad \operatorname{res}_{\lambda=-k} P^\lambda = \frac{(-1)^k}{(k-1)!}\delta_1^{(k-1)}(P) \tag{1.2}$$

for $p+q = n$ is odd with p odd and q even. Also, for the singular point $\lambda = -n/2-k$ they obtained

$$\operatorname{res}_{\lambda=-n/2-k} P^\lambda = \frac{(-1)^{q/2} L^k \delta(x)}{2^{2k} k!\,\Gamma((n/2)+k)} \tag{1.3}$$

for $p+q = n$ is odd with p odd and q even.

Now, let $K_{\alpha,\beta}(x)$ be the convolution of the functions $R_\alpha^H(u)$ and $R_\beta^\ell(v)$, that is,

$$K_{\alpha,\beta}(x) = R_\alpha^H(u) * R_\beta^\ell(v), \tag{1.4}$$

where $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are defined by (2.1) and (2.3), respectively. Since $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are tempered distributions, see [4, pages 30–31], thus $K_{\alpha,\beta}(x)$ is also a tempered distribution and is called the distributional kernel.

In this paper, we use the idea of Gel'fand and Shilov to find the residue of the Fourier transform $(-1)^k\widehat{K_{2k,2k}(\xi)}$, where $K_{2k,2k}$ is defined by (1.4) with $\alpha = \beta = 2k$ and k is a nonnegative integer. We found that for any nonzero point ξ that belongs to the spectrum of $(-1)^k K_{2k,2k}(x)$, there exists the residue of the Fourier transform

$(-1)^k\widehat{K_{2k,2k}(\xi)}$. Actually $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator $\diamond^k$ iterated k times, that is, $\diamond^k[(-1)^k K_{2k,2k}(x)] = \delta$, where δ is the Dirac-delta distribution.

The operator $\diamond^k$ was first introduced by Kananthai [4] and named as the Diamond operator defined by

$$\diamond^k = \left[\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2}+\frac{\partial^2}{\partial x_{p+2}^2}+\cdots+\frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k, \tag{1.5}$$

where $p+q=n$ is the dimension of $\mathbb{R}^n$.

Moreover, the operator $\diamond^k$ can be expressed as the product of the operators $\square^k$ and $\triangle^k$, that is,

$$\diamond^k = \square^k \triangle^k = \triangle^k \square^k, \tag{1.6}$$

where $\square^k$ is an ultra-hyperbolic operator iterated k times defined by

$$\square^k = \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k, \tag{1.7}$$

where $p+q=n$. The operator $\triangle^k$ is an elliptic operator or Laplacian iterated k times defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)^k. \tag{1.8}$$

Trione [7, page 11] has shown that the function $R_{2k}^H(u)$ defined by (2.1) with $\alpha = 2k$ is an elementary solution of the operator $\square^k$. Also, Aguirre Téllez [1, pages 147–148] has proved that the solution $R_{2k}^H(u)$ exists only for odd n with p odd and q even $(p+q=n)$. Moreover, we can show that the function $(-1)^k R_{2k}^{\ell}(v)$ is an elementary solution of the operator $\triangle^k$, where $R_{2k}^{\ell}(v)$ is defined by (2.3) with $\beta = 2k$.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of $\mathbb{R}^n$, and write $u = x_1^2 + x_2^2+\cdots+x_p^2 - x_{p+1}^2-\cdots-x_{p+q}^2$, $p+q=n$. Denote by $\Gamma_+ = \{x\in\mathbb{R}^n : x_1 > 0,\ u > 0\}$ the set of an interior of the forward cone, and $\overline{\Gamma_+}$ denotes the closure of Γ_+. For any complex number α, define

$$R_\alpha^H(u) = \begin{cases} \dfrac{u^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \tag{2.1}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}{\Gamma((2+\alpha-p)/2)\Gamma((p-\alpha)/2)}. \tag{2.2}$$

The function $R_\alpha^H(u)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Nozaki [6, page 72]. The function R_α^H is an ordinary function or classical function if $\operatorname{Re}(\alpha) \ge n$ and is a distribution of α if $\operatorname{Re}(\alpha) < n$. Let $\operatorname{supp} R_\alpha^H(u) \subset \overline{\Gamma_+}$, where $\operatorname{supp} R_\alpha^H(u)$ denotes the support of $R_\alpha^H(u)$.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$, and write $v = x_1^2 + x_2^2 + \cdots + x_n^2$. For any complex number β, define

$$R_\beta^\ell(v) = \frac{2^{-\beta}\pi^{-n/2}\Gamma((n-\beta)/2)v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \tag{2.3}$$

The function $R_\beta^\ell(v)$ is called the elliptic kernel of Marcel Riesz and is an ordinary function for $\mathrm{Re}(\beta) \geq n$ and is a distribution of β for $\mathrm{Re}(\beta) < n$.

Definition 2.3. Let f be a continuous function, then the Fourier transform of f, denoted by $\mathfrak{I}f$ or $\hat{f}(\xi)$, is defined by

$$\mathfrak{I}f = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\,dx, \tag{2.4}$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, and $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$. From (2.4), the inverse Fourier transform of $\hat{f}(\xi)$ is defined by

$$f(x) = \mathfrak{I}^{-1}\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi)\,dx. \tag{2.5}$$

If f is a distribution with compact support, by [8, Theorem 7.4.3, page 187] (2.5) can be written as

$$\mathfrak{I}f = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \langle f(x), e^{-i(\xi,x)} \rangle. \tag{2.6}$$

Lemma 2.4. *Given the equation*

$$\diamond^k u(x) = \delta, \tag{2.7}$$

where $\diamond^k$ is the operator defined by (1.5), and δ is the Dirac-delta distribution, $u(x)$ is an unknown, k is a nonnegative integer and $x \in \mathbb{R}^n$, where n is odd with p odd, q even $(n = p+q)$. Then $u(x) = (-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator $\diamond^k$. Here $K_{2k,2k}(x) = R_{2k}^H(u) * R_{2k}^\ell(v)$ from (1.4) with $\alpha = \beta = 2k$.

Proof. See [4, page 33]. □

In this paper, we study the spectrum of $(-1)^k K_{2k,2k}(x)$, relate to the residue of the Fourier transform $(-1)^k \widehat{K_{2k,2k}(\xi)}$.

Lemma 2.5. *The Fourier transform*

$$\begin{aligned}
\widehat{K_{\alpha,\beta}(\xi)} &= (2\pi)^{n/2}\mathfrak{I}R_\alpha^H(u)\mathfrak{I}R_\beta^\ell(v) \\
&= \frac{(i)^q 2^{\alpha+\beta}\pi^n}{(2\pi)^{n/2}K_n(\alpha)H_n(\beta)} \cdot \frac{\Gamma(\alpha/2)\Gamma(\beta/2)}{\Gamma((n-\alpha)/2)\Gamma((n-\beta)/2)} \\
&\quad \times \left(\sqrt{\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2}\right)^{-\alpha} \left(\sqrt{\sum_{i=1}^{n}\xi_i^2}\right)^{-\beta}, \quad i = \sqrt{-1}.
\end{aligned} \tag{2.8}$$

In particular, if $\alpha = \beta = 2k$, k is a nonnegative integer,

$$(-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{1}{(2\pi)^{n/2}} \frac{1}{((\xi_1^2+\xi_2^2+\cdots+\xi_p^2)^2-(\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2)^2)^k}, \quad (2.9)$$

where $R_\alpha^H(u)$ and $R_\beta^\ell(v)$ are defined by (2.1) and (2.3), respectively.

Proof. See [2, page 194] and [5, pages 156–157]. □

Definition 2.6. The spectrum of the distributional kernel $K_{\alpha,\beta}(x)$ is the support of the Fourier transform $\widehat{K_{\alpha,\beta}(\xi)}$ or the spectrum of $K_{\alpha,\beta}(x) = \operatorname{supp}\widehat{K_{\alpha,\beta}(\xi)}$. Now, from Lemma 2.5 we obtain

$$\operatorname{supp}\widehat{K_{\alpha,\beta}(\xi)} = (\operatorname{supp}\Im R_\alpha^H(u)) \cap (\operatorname{supp}\Im R_\beta^\ell(v)). \quad (2.10)$$

In particular, from (2.9) the spectrum of

$$(-1)^k K_{2k,2k}(x) = \operatorname{supp}\left[\frac{1}{(2\pi)^{n/2}((\sum_{i=1}^p \xi_i^2)^2-(\sum_{j=p+1}^{p+q}\xi_j^2)^2)^k}\right]. \quad (2.11)$$

Lemma 2.7. *Let $P(x_1,x_2,\dots,x_n)$ be a quadratic form of positive definite, and is defined by*

$$P = P(x_1,x_2,\dots,x_n) = \left(\sum_{i=1}^p x_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} x_j^2\right)^2, \quad (2.12)$$

then for any testing function $\varphi(x) \in D$, the space of infinitely differentiable function with compact support,

$$\langle \delta^{(k)}(P), \varphi\rangle = \int_0^\infty \left[\left(\frac{\partial}{4s^3\partial s}\right)^k \left(s^{q-4}\frac{\psi(r,s)}{4}\right)\right]_{s=r} r^{p-1}\,dr, \quad (2.13)$$

$$\langle \delta^{(k)}(P), \varphi\rangle = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{4r^3\partial r}\right)^k \left(r^{p-4}\frac{\psi(r,s)}{4}\right)\right]_{r=s} s^{q-1}\,ds, \quad (2.14)$$

where $r^2 = x_1^2+x_2^2+\cdots+x_p^2$, $s^2 = x_{p+1}^2+x_{p+2}^2+\cdots+x_{p+q}^2$, and

$$\psi(r,s) = \int \varphi\, d\Omega^p\, d\Omega^q, \quad (2.15)$$

where $d\Omega^p$ and $d\Omega^q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Both integrals (2.13) and (2.14) converge if $k < (1/4)(p+q-4)$ for any $\varphi(x) \in D$. If $k \ge (1/4)(p+q-4)$, these integrals must be understood in the sense of their regularization and (2.13) defined as $\langle \delta_1^{(k)}(p), \varphi\rangle$ and (2.14) defined as $\langle \delta_2^{(k)}(p), \varphi\rangle$. Moreover, if we put $u = r^2$, $v = s^2$, thus (2.13) and (2.14) become

$$\langle \delta^{(k)}(p), \varphi\rangle = \frac{1}{16}\int_0^\infty \left[\frac{\partial^k}{\partial v^k}\left(v^{(q-4)/4}\psi_1(u,v)\right)\right]_{v=u} u^{(1/4)(p-4)}\,du, \quad (2.16)$$

$$\langle \delta^{(k)}(p), \varphi\rangle = \frac{(-1)^k}{16}\int_0^\infty \left[\frac{\partial^k}{\partial u^k}\left(u^{(p-4)/4}\psi_1(u,v)\right)\right]_{u=v} v^{(1/4)(q-4)}\,dv, \quad (2.17)$$

where $\psi_1(u,v) = \psi(r,s)$.

Proof. See [2, pages 247–251]. □

Lemma 2.8. *Let* $G_b = \{\xi \in \mathbb{R}^n : |\xi_1| \le b_1, |\xi_2| \le b_2, \dots, |\xi_n| \le b_n\}$ *be a parallelepiped in* $\mathbb{R}^n$ *and* b_i $(1 \le i \le n)$ *is a real constant and the inverse Fourier transform of* $\widehat{K_{\alpha,\beta}(\xi)}$ *is defined by*

$$K_{\alpha,\beta}(x) = \mathfrak{I}^{-1}\widehat{K_{\alpha,\beta}(\xi)} = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i(\xi,x)} \widehat{K_{\alpha,\beta}(\xi)} d\xi, \tag{2.18}$$

where $K_{\alpha,\beta}$ *is defined by (1.4) and* $x, \xi \in \mathbb{R}^n$, *then* $K_{\alpha,\beta}(x)$ *can be extended to the entire function* $K_{\alpha,\beta}(z)$ *and be analytic for all* $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, *where* $\mathbb{C}^n$ *is the* n*-tuple space of complex number and*

$$|K_{\alpha,\beta}(z)| \le C \exp(b|\operatorname{Im}(z)|), \tag{2.19}$$

where $\exp(b|\operatorname{Im}(z)|) = \exp[b_1|\operatorname{Im}(z_1)| + b_2|\operatorname{Im}(z_2)| + \cdots + b_n|\operatorname{Im}(z_n)|]$ *and* $C = (1/(2\pi)^{n/2}) \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi$ *is a constant. Moreover,* $K_{\alpha,\beta}(x)$ *has a spectrum contained in* G_b.

Proof. Since the integral of (2.18) converges for all $\xi \in G_b$, thus $K_{\alpha,\beta}(x)$ can be extended to the entire function $K_{\alpha,\beta}(z)$ and be analytic for all $z \in C^n$. Thus (2.18) can be written as

$$K_{\alpha,\beta}(z) = \frac{1}{(2\pi)^{n/2}} \int_{G_b} e^{i(\xi,z)} \widehat{K_{\alpha,\beta}(\xi)} d\xi. \tag{2.20}$$

Now,

$$\begin{aligned} |K_{\alpha,\beta}(z)| &\le \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| \, |\exp(i\xi_1 z_1 + i\xi_2 z_2 + \cdots + i\xi_n z_n)| d\xi \\ &= \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| \, |\exp(i\xi_1\sigma_1 + i\xi_2\sigma_2 + \cdots + i\xi_n\sigma_n \\ &\qquad - \xi_1\mu_1 - \xi_2\mu_2 - \cdots - \xi_n\mu_n)| d\xi, \end{aligned} \tag{2.21}$$

where

$$z_j = \sigma + i\mu_j \quad (j = 1, 2, \dots, n), \tag{2.22}$$

thus

$$|K_{\alpha,\beta}(z)| \le \frac{1}{(2\pi)^{n/2}} \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi \exp(b_1|\mu_1| + b_2|\mu_2| + \cdots + b_n|\mu_n|) \tag{2.23}$$

for $|\xi_j| \le b_j$, or $|K_{\alpha,\beta}(z)| \le C \exp(b_1|\operatorname{Im}(z_1)| + b_2|\operatorname{Im}(z_2)| + \cdots + b_n|\operatorname{Im}(z_n)|)$, or $|K_{\alpha,\beta}(z)| \le C \exp(b|\operatorname{Im}(z)|)$, where $C = (1/(2\pi)^{n/2}) \int_{G_b} |\widehat{K_{\alpha,\beta}(\xi)}| d\xi$ is a constant. □

We must show that the support of $\widehat{K_{\alpha,\beta}(\xi)}$ is contained in G_b. Since $K_{\alpha,\beta}(z)$ is an analytic function that satisfies the inequality (2.19) and is called an entire function of order of growth ≤ 1 and of type $\le b$, then by Paley-Wiener-Schartz theorem, see [3, page 162], $\widehat{K_{\alpha,\beta}(\xi)}$ has a support contained in G_b, that is the spectrum of $K_{\alpha,\beta}(x)$ is contained in G_b.

In particular, for $\alpha = \beta = 2k$, the spectrum of $(-1)^k K_{2k,2k}(x)$ is also contained in G_b, that is $\operatorname{supp}[(-1)^k \widehat{K_{2k,2k}(\xi)}] \subset G_b$, where $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the Diamond operator $\diamond^k$ by Lemma 2.4, and the Fourier transform $(-1)^k \widehat{K_{2k,2k}(\xi)}$ given by (2.9) can be defined as follows.

Definition 2.9. The Fourier transform

$$(-1)^k \widehat{K_{2k,2k}(\xi)} = \begin{cases} \dfrac{1}{(2\pi)^{n/2}\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]^k}, & \text{for } \xi \in G_b, \\ 0, & \text{for } \xi \in CG_b, \end{cases} \tag{2.24}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ and CG_b is the complement of G_b.

3. Main results

Theorem 3.1. *For any nonzero point $\xi \in M$ where M is a spectrum of $(-1)^k K_{2k,2k}(x)$, and $(-1)^k K_{2k,2k}(x)$ is an elementary solution of the operator $\diamond^k$ by Lemma 2.4. Then there exists the residue of the Fourier transform $(-1)^k \widehat{K_{2k,2k}(\xi)}$ at the singular point $\lambda = -k$ and such a residue is*

$$\frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!}\delta_1^{(k-1)(p)} \quad \text{or} \quad \operatorname{res}_{\lambda=-k}(-1)^k \widehat{K_{2k,2k}(\xi)} = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!}\delta^{(k-1)(p)}, \tag{3.1}$$

where

$$P = \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right), \tag{3.2}$$

$p + q = n$ and $\delta_1^{(k-1)}(P)$ is defined by (2.16) with $\delta^{(k-1)}(P) = \delta_1^{(k-1)}(P)$ and n is odd with p odd, q even.

Proof. We define the generalized function P^λ, where P is given by (3.2) and λ is a complex number, by

$$\langle P^\lambda, \varphi\rangle = \int_{P>0} P^\lambda(\xi)\varphi(\xi)\,d\xi, \tag{3.3}$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $d\xi = d\xi_1 d\xi_2 \cdots d\xi_n$ and $\varphi(\xi) \in D$, the space of continuous infinitely differentiable function with compact support. Now,

$$\langle P^\lambda, \varphi\rangle = \int_{P>0} \left[\left(\xi_1^2 + \xi_2^2 + \cdots + \xi_p^2\right)^2 - \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2\right)\right]^\lambda \varphi(\xi)\,d\xi. \tag{3.4}$$

We transform to bipolar coordinates defined by

$$\begin{gathered} \xi_1 = r w_1,\ \xi_2 = r w_2,\ \dots,\ \xi_p = r w_p, \\ \xi_{p+1} = s w_{p+1},\ \xi_{p+2} = s w_{p+2},\ \dots,\ \xi_{p+q} = s w_{p+q}, \quad p + q = n, \end{gathered} \tag{3.5}$$

where $\sum_{i=1}^{p} w_i^2 = 1$ and $\sum_{j=p+1}^{p+q} w_j^2 = 1$. Thus

$$r = \sqrt{\sum_{i=1}^{p}\xi_i^2}, \qquad s = \sqrt{\sum_{j=p+1}^{p+q}\xi_j^2}. \tag{3.6}$$

We have $\langle P^\lambda,\varphi\rangle = \int [r^4 - s^4]^\lambda \varphi(\xi)d\xi$. Since the volume $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$ where $d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Thus

$$\begin{aligned}\langle P^\lambda,\varphi\rangle &= \int_{p>0} (r^4 - s^4)^\lambda \varphi r^{p-1}s^{q-1}\,dr\,ds\,d\Omega^p\,d\Omega^q \\ &= \int_0^\infty \int_0^r (r^4 - s^4)^\lambda \psi(r,s) r^{p-1}s^{q-1}\,ds\,dr,\end{aligned} \tag{3.7}$$

where $\psi(r,s) = \int \varphi\, d\Omega_p\, d\Omega_q$.

Since $\varphi(\xi)$ is in D, then $\psi(r,s)$ is an infinitely differentiable function of r^4 and s^4 with bounded support. We now make the change of variable $u = r^4$, $v = s^4$, and writing $\psi(r,s) = \psi_1(u,v)$. Thus we obtain

$$\langle P^\lambda,\varphi\rangle = \frac{1}{16}\int_{u=0}^\infty \int_{v=0}^u (u-v)^\lambda \psi_1(u,v) u^{(p-4)/4} v^{(q-4)/4}\,dv\,du. \tag{3.8}$$

Write $v = ut$. We obtain

$$\langle P^\lambda,\varphi\rangle = \frac{1}{16}\int_0^\infty u^{\lambda+(1/4)(p+q)-1}\,du \int_0^1 (1-t)^\lambda t^{(q-4)/4}\psi_1(u,ut)dt. \tag{3.9}$$

Let the function

$$\Phi(\lambda,u) = \frac{1}{16}\int_0^1 (1-t)^\lambda t^{(q-4)/4}\psi_1(u,ut)dt. \tag{3.10}$$

Thus $\Phi(\lambda,u)$ has singularity at $\lambda = -k$ where it has simple poles. By Gel'fand and Shilov [2, page 254, equation (12)] we obtain the residue of $\Phi(\lambda,u)$ at $\lambda = -k$, that is,

$$\operatorname{res}_{\lambda=-k}\Phi(\lambda,u) = \frac{1}{16}\frac{(-1)^{k-1}}{(k-1)!}\left[\frac{\partial^{k-1}}{\partial t^{k-1}}\{t^{(q-4)/4}\psi_1(u,ut)\}\right]_{t=1}. \tag{3.11}$$

Thus, $\operatorname{res}_{\lambda=-k}\Phi(\lambda,u)$ is a functional concentrated on the surface $P = 0$ ($t = 1$, $u = v$, $p = u - v = 0$). On the other hand, from (3.9) and (3.10) we have

$$\langle P^\lambda,\varphi\rangle = \int_0^\infty u^{\lambda+(1/4)(p+q)-1}\Phi(\lambda,u)du. \tag{3.12}$$

Thus $\langle P^\lambda,\varphi\rangle$ in (3.12) has singularities at $\lambda = -n/4, -n/4-1, \ldots, -n/4-k$. At these points,

$$\operatorname{res}_{\lambda=-n/4-k}\langle P^\lambda,\varphi\rangle = \frac{1}{k!}\left[\frac{\partial^k}{\partial u^k}\Phi\left(-\frac{n}{4}-k,u\right)\right]_{u=0}. \tag{3.13}$$

Thus the residue of $\langle P^\lambda,\varphi\rangle$ at $\lambda = (-1/2)n - k$ is a functional concentrated on the vertex of the surface P. Now consider the case when the singular point $\lambda = -k$. Write (3.10) in the neighborhood of $\lambda = -k$ in the form $\Phi(\lambda,u) = \Phi_0(u)/(\lambda+k) + \Phi_1(\lambda,u)$ where $\Phi_0(u) = \operatorname{res}_{\lambda=-k}\Phi(\lambda,u)$ and $\Phi_1(\lambda,u)$ is regular at $\lambda = -k$. Substitute $\Phi(\lambda,u)$ into (3.12) we obtain

$$\langle P^\lambda,\varphi\rangle = \frac{1}{\lambda+k}\int_0^\infty u^{\lambda+(1/4)(p+q)-1}\Phi_0(u)du + \int_0^\infty u^{\lambda+(1/4)(p+q)-1}\Phi_1(\lambda,u)du. \tag{3.14}$$

Thus $\operatorname{res}_{\lambda=-k}\langle P^\lambda,\varphi\rangle = \int_0^\infty u^{-k+(1/4)(p+q)-1}\Phi_0(u)\,du$. By substituting $\Phi_0(u)$ and (3.11), we obtain

$$\operatorname{res}_{\lambda=-k}\langle P^\lambda,\varphi\rangle = \frac{(-1)^k}{16(k-1)!}\int_0^\infty\left[\frac{\partial^{k-1}}{\partial t^{k-1}}\{t^{1(q-4)/4}\psi_1(u,ut)\}\right]_{t=1}u^{-k+(1/4)(p+q)-1}\,du \tag{3.15}$$

since, we put $v = ut$. Thus $\partial^{k-1}/\partial t^{k-1} = u^{k-1}(\partial^{k-1}/\partial v^{k-1})$, by substituting $\partial^{k-1}/\partial t^{k-1}$ we obtain

$$\operatorname{res}_{\lambda=-k}\langle P^\lambda,\varphi\rangle = \frac{(-1)^k}{16(k-1)!}\int_0^\infty\left[\frac{\partial^{k-1}}{\partial t^{k-1}}\{v^{1(q-4)/4}\psi_1(u,v)\}\right]_{u=v}u^{(1/4)p-1}\,du. \tag{3.16}$$

Now, by (2.16)

$$\operatorname{res}_{\lambda=-k}\langle P^\lambda,\varphi\rangle = \frac{(-1)^{k-1}}{(k-1)!}\delta_1^{(k-1)}(P). \tag{3.17}$$

Since, by Definition 2.9 we have

$$(-1)^k\widehat{K_{2k,2k}(\xi)} = \frac{1}{(2\pi)^{n/2}}P^\lambda \quad \text{for } \lambda = -k, \tag{3.18}$$

and $\xi \in G_b$. Let M be a spectrum of $(-1)^kK_{2k,2k}(x)$ and $M \subset G_b$ by Lemma 2.8. Thus for any nonzero $\xi \in M$ we can find the residue of $(-1)^k\widehat{K_{2k,2k}(\xi)}$, that is,

$$\begin{aligned}\operatorname{res}_{\lambda=-k}\langle(-1)^k\widehat{K_{2k,2k}(\xi)},\varphi(\xi)\rangle &= \frac{1}{(2\pi)^{n/2}}\operatorname{res}_{\lambda=-k}\langle P^\lambda,\varphi\rangle\\ &= \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!}\langle\delta_1^{(k-1)}(P),\varphi\rangle\end{aligned} \tag{3.19}$$

or $\operatorname{res}_{\lambda=-k}(-1)^k\widehat{K_{2k,2k}(\xi)} = ((-1)^{k-1}/(2(\pi)^{n/2}(k-1)!))\delta_1^{(k-1)}(P)$ for $\xi \in M$ and $\xi \neq 0$.

Now consider the case $\xi = 0$. We have from (3.13) that, the residue of $\langle P^\lambda,\varphi\rangle$ occurs at the point $\lambda = (-1/2)n-k$ that is $\operatorname{res}_{\lambda=-(1/2)n-k}\langle P^\lambda,\varphi\rangle$ is a functional concentrated on the vertex of surface P. Since $u = 0$ and $v = ut$, then $u = v = 0$, that implies

$$\sqrt{\xi_1^2+\xi_2^2+\cdots+\xi_p^2} = \sqrt{\xi_{p+1}^2+\xi_{p+2}^2+\cdots+\xi_{p+q}^2} = 0. \tag{3.20}$$

It follows that $\xi_1 = \xi_2 = \cdots = \xi_{p+q} = 0$, $p+q = n$. Thus, the residue of $\langle P^\lambda,\varphi\rangle$ is concentrated on the point $\xi = 0$.

Since, from Definition 2.9, $(1/(2\pi)^{n/2})P^\lambda = (-1)^k\widehat{K_{2k,2k}(\xi)}$ if $\lambda = -k$. Thus we only consider the residue of $(-1)^k\widehat{K_{2k,2k}(\xi)}$ at $\lambda = -k$. From (3.12), we consider the residue of $\langle P^\lambda,\varphi\rangle$ only at $\lambda = -k$. That implies $(1/4)(p+q)-1 = 0$ or $n = 4$ $(p+q = n)$. Since $n = 4$ is an even dimension which contradicts Lemma 2.4, the existence of the elementary solution $(-1)^kK_{2k,2k}(x)$ that exists for odd n. Thus cases (3.12) and (3.13) do not occur. This implies that the case $\xi = 0$ does not happen. It follows that

$$\operatorname{res}_{\lambda=-k}(-1)^k\widehat{K_{2k,2k}(\xi)} = \frac{(-1)^{k-1}}{(2\pi)^{n/2}(k-1)!}\delta_1^{(k-1)}(P) \tag{3.21}$$

for nonzero point $\xi \in M$ concentrated on the surface $P = 0$, where M is a spectrum of $(-1)^kK_{2k,2k}(x)$. □

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