QUATERNION CR-SUBMANIFOLDS OF A QUATERNION KAEHLER MANIFOLD

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ABSTRACT. We study the quaternion CR-submanifolds of a quaternion Kaehler manifold. More specifically we study the properties of the canonical structures and the geometry of the canonical foliations by using the Bott connection and the index of a quaternion CR-submanifold.

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1. Introduction. The notion of a CR-submanifold of a Kaehler manifold was introduced by Bejancu [3]. Subsequently a number of authors studied these submanifolds (see [4] for details). In [1], Barros et al. studied quaternion CR-submanifolds of a quaternion Kaehler manifold and obtained many interesting results. The aim of this paper is to continue the study of quaternion CR-submanifolds of a quaternion Kaehler manifold. The paper is organized as follows: in Section 2 we collect some basic formulas and results for later use and in Section 3 we study some properties of canonical structures, particularly its parallelism and QR-product. In Section 4 we study the geometry of the canonical foliations using the Bott connection and the index of a quaternion CR-submanifold. Finally, as an extension of the work of Chen [5] for the Kaehler manifolds we give a complete classification of the totally umbilical quaternion CR-submanifolds of a quaternion Kaehler manifold.

2. Preliminaries. Let $\bar{M}$ be a quaternion Kaehler manifold with metric tensor $g$ and quaternion structure $V$ [7]. We will denote by $\psi_1 = I$, $\psi_2 = J$, and $\psi_3 = K$ a local basis of almost Hermitian structures for $V$.

Let $X$ be a unit vector tangent to the quaternion Kaehler manifold $\bar{M}$. Then the vectors $X, IX, JX, KX$ form an orthonormal frame. Let $Q(X)$ be the quaternion section determined by $X$. Any plane in a quaternion section is called a quaternion plane and the sectional curvature of a quaternion plane is called a quaternion sectional curvature. A quaternion Kaehler manifold is called a quaternion space form, which is denoted by $\bar{M}(c)$, if its quaternion sectional curvature is equal to a constant $c$ at any point of the manifold. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by, [7],

$$\bar{R}(X, Y)Z = \frac{c}{4} \left[ g(Y, Z)X - g(X, Z)Y + \sum_{r=1}^{3} g(\psi_r Y, Z)\psi_r X - g(\psi_r X, Z)\psi_r Y + 2g(X, \psi_r Y)\psi_r Z \right],$$

(2.1)

where $\psi_1 = I$, $\psi_2 = J$, $\psi_3 = K$. 
Let $M$ be a Riemannian manifold isometrically immersed in a quaternion Kaehler manifold $\bar{M}$. We also denote by $g$ the metric tensor induced on $M$. If $\nabla$ is the covariant differentiation induced on $M$, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$  

(2.2)

respectively, for any $X, Y$ tangent to $M$ and $N$ normal to $M$. Here $h$ and $\nabla^\perp$ are the second fundamental form associated with $M$, and the connection of the normal bundle, respectively. The second fundamental tensor $A_N$ is related to $h$ by

$$g(A_N X, Y) = g(h(X, Y), N).$$

(2.3)

A differentiable distribution $D_x$ on $M$ such that $\psi_r(D_x) \subseteq D_x$ for all $r = 1, 2, 3$ is called a quaternion distribution. In other words, $D_x$ is a quaternion distribution if $D_x$ is contained into itself by its quaternion structure.

It is known [1] that a submanifold $M$ of a quaternion Kaehler manifold $\bar{M}$ is called a quaternion CR-submanifold if it admits a quaternion distribution $D_x$ such that its orthogonal complementary distribution $D_\perp x$, is totally real, that is, $\psi_r(D_\perp x) \subseteq T^\perp x M$ for all $x \in M$ and $r = 1, 2, 3$, where $T^\perp_x M$ denotes the normal space of $M$ at $x$.

A submanifold $M$ of a quaternion Kaehler manifold $\bar{M}$ is called a quaternion (resp., totally real) submanifold if $\dim D_\perp x = 0$ (resp., $\dim D_x = 0$). A quaternion CR-submanifold is said to be proper if it is neither quaternion nor totally real.

We denote by $\mu$ the subbundle of the normal bundle $T^\perp M$ which is the orthogonal complement of $\psi_1 D_\perp \oplus \psi_2 D_\perp \oplus \psi_3 D_\perp$, that is,

$$T^\perp M = \psi_1 D_\perp \oplus \psi_2 D_\perp \oplus \psi_3 D_\perp \oplus \mu; \quad g(\mu, \psi_r D_\perp) = 0.$$  

(2.4)

The mean curvature vector $H$ of $M$ in $\bar{M}$ is defined by $H = (1/n) \text{trace} h$, where $n$ denotes the dimension of $M$. If we have

$$h(X, Y) = g(X, Y) H$$

(2.5)

for any $X, Y \in TM$, then $M$ is called a totally umbilical submanifold. In particular, if $h(X, Y) = 0$ identically for all $X, Y \in TM$, $M$ is called a totally geodesic submanifold. Finally $M$ is called mixed totally geodesic if $h(X, Y) = 0$ for $X \in D, Y \in D^\perp$. For totally umbilical CR-submanifolds, equations (2.2) take the forms

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, Y) H, \quad \bar{\nabla}_X N = -g(H, N) X + \nabla^\perp_X N.$$  

(2.6)

The Codazzi equation for a totally umbilical CR-submanifold $M$, is given by

$$\bar{R}(X, Y; Z, N) = g(Y, Z) g(\nabla^\perp_X H, N) - g(X, Z) g(\nabla^\perp_Y H, N).$$  

(2.7)

**Definition 2.1** (see [1]). Let $M$ be a quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then $M$ is called a QR-product if $M$ is locally the Riemannian product of a quaternion submanifold and a totally real submanifold of $\bar{M}$. 


For any $X \in TM$ and $N \in T^\perp M$, we put
\begin{align}
\psi_r X &= P_r X + Q_r X, \\
\psi_r N &= t_r N + f_r N,
\end{align}
where $P_r X$, $t_r N$ (resp., $Q_r X$, $f_r N$) are the tangential (resp., the normal) components of $\psi_r X$ and $\psi_r N$ for $r = 1, 2, 3$.

For the second fundamental form $h$, the covariant differentiation is defined by
\begin{align}
(\tilde{\nabla}_X h)(Y, Z) &= \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \\
R(X, Y, Z, W) &= \tilde{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \\
[R(X, Y) Z]^\perp &= (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z), \quad \forall X, Y, Z, W \text{ tangent to } \tilde{M},
\end{align}
where $R$ is the curvature tensor associated with $\nabla$ and $\perp$ in (2.12) denotes the normal component.

We collect from Barros et al. [1] the following results which we shall need in the sequel.

**Lemma 2.2.** Every quaternion submanifold of a quaternion Kaehler manifold is totally geodesic.

**Lemma 2.3.** The quaternion distribution $D$ of a quaternion CR-submanifold $M$ in a quaternion Kaehler manifold $\tilde{M}$ is integrable if and only if $h(D, D) = 0$.

**Lemma 2.4.** Let $M$ be a quaternion CR-submanifold of a quaternion Kaehler manifold $\tilde{M}$. Then the leaf $M^\perp$ of $D^\perp$ is totally geodesic in $M$ if and only if $g(h(D, D^\perp), \psi_r D^\perp) = 0$, $r = 1, 2, 3$.

**Lemma 2.5.** Let $M$ be a quaternion CR-submanifold of a quaternion Kaehler manifold $\tilde{M}$. Then
\[ A_{\psi_r W} Z = A_{\psi_r Z} W \quad \text{for any } W, Z \in D^\perp. \]  

**3. Canonical parallel structures and QR-product.** Let $P_r, f_r, Q_r$, and $t_r$ be the endomorphisms and the vector-bundle-valued 1-forms defined in (2.8), respectively. We define the covariant differentiation of $P_r, Q_r, t_r$, and $f_r$ as follows:
\begin{align}
(\tilde{\nabla}_X P_r)(Y) &= \nabla_X (P_r Y) - P_r \nabla_X Y, \\
(\tilde{\nabla}_X Q_r)(Y) &= \nabla_X (Q_r Y) - Q_r \nabla_X Y, \\
(\tilde{\nabla}_X t_r)(N) &= \nabla_X (t_r N) - t_r \nabla_X N, \\
(\tilde{\nabla}_X f_r)(N) &= \nabla_X (f_r N) - f_r \nabla_X N,
\end{align}
for any vector fields $X, Y \in TM$ and $N \in T^\perp M$.

The endomorphisms $P_r$ (resp., the endomorphisms $f_r$, the 1-forms $Q_r$ and $t_r$) are parallel if $\tilde{\nabla} P_r = 0$ (resp., $\tilde{\nabla} f_r = 0$, $\tilde{\nabla} Q_r = 0$, and $\tilde{\nabla} t_r = 0$).
Now using the definition of a quaternion Kaehler manifold and taking account of (2.2), (2.8), we can easily obtain the following:

\[
(\hat{\nabla}_X P_r)(Y) = A_{Q_r}X + t_r h(X,Y), \tag{3.2}
\]
\[
(\hat{\nabla}_X Q_r)(Y) = f_r h(X,Y) - h(X, P_r Y), \tag{3.3}
\]
\[
(\hat{\nabla}_X t_r)(N) = A_{f_r}N X - P_r A_N X, \tag{3.4}
\]
\[
(\hat{\nabla}_X f_r)(N) = - h(X, t_r N), \tag{3.5}
\]
for any \(X, Y \in TM\) and \(N \in T^\perp M\).

**Remark 3.1.** Since the second fundamental form is symmetric, it follows from (3.2) that \(P_r\) is parallel if and only if
\[
A_{\psi_r} U V = A_{\psi_r} V U, \quad \forall U, V \in TM. \tag{3.6}
\]

Now if we set \(V = X \in D\) in (3.6), we find that \(A_{\psi_r} U X = 0\) for all \(U \in TM\), which is equivalent to \(g(h(X, Y), \psi_r U) = 0\) for any \(X \in D\), and \(Y, U \in TM\). In particular \(g(h(X, Y), \psi_r Z) = 0\) for any \(X \in D\) and \(Y, Z \in D^\perp\).

Thus, using Lemma 2.4 we obtain the following lemma.

**Lemma 3.2.** Let \(M\) be a quaternion CR-submanifold of a quaternion Kaehler manifold \(\bar{M}\). If \(P_r\) is parallel then the leaf \(M^\perp\) of \(D^\perp\) is totally geodesic in \(M\).

Now we state and prove the following proposition.

**Proposition 3.3.** Let \(M\) be a quaternion CR-submanifold of a quaternion Kaehler manifold \(\bar{M}\). Then \(Q_r\) is parallel if and only if \(t_r\) is parallel.

**Proof.** Suppose \(t_r\) is parallel. Then from (3.4) we have
\[
A_{f_r} U N = P_r A_N U, \quad \text{for any } U \in TM. \tag{3.7}
\]
Thus for any vector fields \(U, V \in TM\) and \(N \in T^\perp M\), we get
\[
g(A_{f_r} U N, V) = g(P_r A_N U, V), \tag{3.8}
\]
or equivalently
\[
f_r h(U, V) - h(U, P_r V) = 0, \tag{3.9}
\]
that is, \(\hat{\nabla} Q_r = 0\).

The proof of the converse statement is similar. \(\Box\)

**Lemma 3.4.** Let \(M\) be a QR-product of a quaternion Kaehler manifold \(\bar{M}\). Then
(a) \(\nabla_Z X \in D\),
(b) \(\nabla_X Z \in D^\perp\),
for all \(X \in D\) and \(Z \in D^\perp\).
Proof. By using (2.2) and the definition of a quaternion Kaehler manifold, we find

\[ \psi_\tau \nabla ZX = \nabla Z \psi_\tau X + h(Z, \psi_\tau X) - \psi_\tau (X, Z) \quad \text{for } X \in D, Z \in D^\perp. \]  

(3.10)

The above equation yields

\[ g(\psi_\tau \nabla ZX, \psi_\tau W) = g(\nabla Z \psi_\tau X, \psi_\tau W) + g(h(Z, \psi_\tau X), \psi_\tau W), \]
\[ g(\nabla ZX, W) = g(h(Z, \psi_\tau X), \psi_\tau W) \quad \text{for } X \in D, W, Z \in D^\perp. \]  

(3.11)

Since \( M \) is a QR-product the leaf \( M^\perp \) of \( D^\perp \) is totally geodesic. Thus using Lemma 2.4 we get (a).

Next for \( X \in D, Z \in D^\perp \) we have

\[ \hat{\nabla}_X \psi_\tau Z = \psi_\tau \hat{\nabla}_X Z \]  

(3.12)

which, by virtue of (2.2), gives

\[ \psi_\tau \nabla ZX = -A_{\psi_\tau} ZX + \nabla_Z \psi_\tau Z - \psi_\tau h(X, Z). \]  

(3.13)

Taking inner products with \( Y \in D \) and using the fact that the leaf \( M^\perp \) of \( D^\perp \) is totally geodesic, we find

\[ g(\psi_\tau \nabla ZX, Y) = -g(A_{\psi_\tau} ZX, Y) = -g(h(X, Y), \psi_\tau Z) \quad \text{for } X, Y \in D, Z \in D^\perp. \]  

(3.14)

On the other hand, for \( X \in D \) and \( W, Z \in D^\perp \) and the use of Lemma 2.5, (3.13) gives

\[ g(\psi_\tau \nabla ZX, W) = -g(\psi_\tau h(X, Z), W) - g(h(X, W), \psi_\tau Z) \]
\[ = g(A_{\psi_\tau} ZX - A_{\psi_\tau} W, X) \]
\[ = 0. \]  

(3.15)

Thus from (3.14) and (3.15) we see that \( \psi_\tau \nabla ZX \) is normal to \( M \). So \( \nabla ZX \in D^\perp \) for all \( X \in D \) and \( Z \in D^\perp \).

Therefore by virtue of [1, Lemma 5.1, page 403], we get \( h(D, D^\perp) = 0 \) or \( h(D, D) = 0 \). So the quaternion distribution \( D \) is integrable by virtue of Lemma 2.3. Thus it follows that each leaf \( M^\perp \) is totally geodesic in \( \hat{M} \) and in particular \( M^\perp \) is totally geodesic in \( M \) by virtue of Lemma 2.2.

Theorem 3.5. Let \( M \) be a quaternion CR-submanifold of a quaternion Kaehler manifold \( \hat{M} \). Then \( M \) is a QR-product if and only if \( P_\tau \) is parallel.

Proof. Suppose \( P_\tau \) is parallel, then from (3.2), we have

\[ A_{Q_\tau} X + t_\tau h(X, Y) = 0 \quad \forall X, Y \in TM. \]  

(3.16)

If \( Y \in D \), then \( Q_\tau Y = 0 \). Hence (3.16) is reduced to \( t_\tau h(X, Y) = 0 \) for all \( X \in TM, Y \in D \).

Therefore by virtue of [1, Lemma 5.1, page 403], we get \( h(D, D^\perp) = 0 \) or \( h(D, D) = 0 \).

So the quaternion distribution \( D \) is integrable by virtue of Lemma 2.3. Thus it follows that each leaf \( M^\perp \) is totally geodesic in \( \hat{M} \) and in particular \( M^\perp \) is totally geodesic in \( M \) by virtue of Lemma 2.2.
Again from (3.2), we have

\[ A_{\psi_r} W + t_r h(W, Z) = 0 \quad \forall W, Z \in D^\perp. \tag{3.17} \]

So for \( X \in D \), we have

\[ g(A_{\psi_r} W, X) + g(t_r h(W, Z), X) = 0 \tag{3.18} \]

which means

\[ g(h(X, Z), Q_r W) - g(h(W, Z), Q_r X) = 0, \tag{3.19} \]

that is,

\[ g(h(X, Z), Q_r W) = 0 \tag{3.20} \]

or

\[ g(h(D, D^\perp), Q_r D^\perp) = 0. \tag{3.21} \]

Thus using Lemma 2.4, it follows that the leaf \( M^\perp \) of \( D^\perp \) is totally geodesic. Hence \( M \) is a QR-product.

Conversely, let \( M \) be a QR-product. First we show that \( \nabla_U X \in D \) for any \( X \in D \) and \( U \) tangent to \( M \). Since \( M \) is a QR-product, that is, locally a Riemannian product of a quaternion submanifold and a totally real submanifold, it is sufficient to show that \( \nabla_Z X \in D \) for any \( X \in D, Z \in D^\perp \) but this was proved in Lemma 3.4(a). Using this fact, we have

\[ \nabla_U \psi_r X + h(U, \psi_r X) = \psi_r \nabla_U X + \psi_r h(X, U) \tag{3.22} \]

for any \( X \in D, U \) tangent to \( M \), which yields

\[ \psi_r h(U, X) = h(U, \psi_r X), \quad \nabla_U \psi_r X = \psi_r \nabla_U X. \tag{3.23} \]

Thus \( (\nabla_U P_r)(X) = \nabla_U P_r X - P_r \nabla_U X = 0 \), for any \( X \in D \), and \( U \) tangent to \( M \).

Similarly, by using Lemma 3.4(b), it follows that \( \nabla_Z X \in D^\perp \) for any \( Z \in D^\perp \), and \( U \) tangent to \( M \). But since \( M \) is a QR-product, it follows that \( \nabla_X Z \in D^\perp \) for \( U = X \in D \) and \( Z \in D^\perp \).

Thus, we have \( (\nabla_U P_r)(Z) = 0 \) for any \( Z \in D^\perp \), \( U \) tangent to \( M \). Therefore \( \nabla P_r = 0 \), which completes the proof.

**Corollary 3.6.** Let \( M \) be a QR-product of a quaternion Kaehler manifold \( \bar{M} \). Then \( M \) is mixed totally geodesic, that is, \( h(D, D^\perp) = 0 \).

**Remark 3.7.** If \( M \) is a proper QR-product of a quaternion space form \( \bar{M}(c) \), then the ambient manifold \( \bar{M} \) is necessarily a space of zero curvature. Hence there does not exist a proper QR-product of a quaternion space form \( \bar{M}(c) \) with \( c \neq 0 \).

### 4. Canonical foliations and index of a quaternion CR-submanifold

**Definition 4.1** (see [8]). Let \( D \) be a distribution on the Riemannian manifold \( M \), \( D^\perp \) the orthogonal distribution, \( \Pi^\perp : TM \to D^\perp \) the projection and \( \nabla \) the Levi-Civita
connection. Then the second fundamental form of the plane field $D$, is defined by

$$S_{\nabla}(X,Y) = \frac{1}{2} \Pi(\nabla_X Y + \nabla_Y X).$$

(4.1)

The distribution $D$ is called a totally geodesic plane field, if the geodesics tangent to it at one point remain tangent for all their length.

Thus we say that the distribution $D$ is a totally geodesic plane field if

$$S_{\nabla}(X,Y) = \Pi(\nabla_X Y + \nabla_Y X) = 0 \quad \forall X,Y \in D.$$

(4.2)

A geometric definition of this notion is given in [9].

A foliation $f$ on a Riemannian manifold $M$ is called a Riemannian foliation, if the Bott connection $\nabla^{\parallel} = \nabla$ in the normal bundle of $f$ preserves the Riemannian metric. Also $f$ is a Riemannian foliation if and only if the second fundamental form $S_{\nabla}$ of the plane field $D$ vanishes (see [9, page 157]).

**THEOREM 4.2.** Let $M$ be a quaternion CR-submanifold of a quaternion Kaehler manifold $\tilde{M}$ such that $D_{\tilde{M}}$ is a totally real foliation of $M$. Then the Bott connection of $D_{\tilde{M}}$ preserves the volume form $\psi$ of $D_{\tilde{M}}$, that is, $\nabla_Z \psi = 0$, for all $Z \in D_{\tilde{M}}$.

**PROOF.** For any $X,Y \in D$ and $Z \in D^\perp$, we have

$$g\left(\left(\nabla_Z \psi_r\right)(X,Y)\right) = g\left(\nabla_Z \psi_r X, Y\right) - g\left(\psi_r \nabla_Z X, Y\right)$$

$$= g\left(\left[[Z,\psi_r X], Y\right] + g\left(\left[Z,X\right],\psi_r Y\right)\right)$$

$$= g\left(\nabla_Z \psi_r X, Y\right) - g\left(\nabla_{\psi_r X} Z, Y\right)$$

$$+ g\left(\nabla_Z X, \psi_r Y\right) - g\left(\nabla_X Z, \psi_r Y\right)$$

$$= g\left(\nabla_Z \psi_r X, Y\right) + g\left(\nabla_{\psi_r X} Z, Y\right)$$

$$- g\left(\nabla_X, \nabla_Z \psi_r Y\right) + g\left(\nabla_{\psi_r Z} Y, \psi_r X\right)$$

$$= g\left(\nabla_{\psi_r Z} Y, Z\right) + g\left(\nabla_X \psi_r Z, Y\right)$$

$$= g\left(\nabla_X \psi_r Z, Y\right) - g\left(A_{\psi_r Z} X, Y\right).$$

(4.3)

Also,

$$g(\nabla_X X, Z) = g(\nabla_X Z)$$

$$= g(\psi_r \nabla_X X, \psi_r Z)$$

$$= g(\nabla_X \psi_r X, \psi_r Z)$$

$$= -g(\nabla_X \psi_r Z, \psi_r X)$$

$$= g(A_{\psi_r Z} X, \psi_r X).$$

(4.4)

If $D_{\tilde{M}}$ is Riemannian then $D_M$ is a totally geodesic plane field and so (4.4) gives $g(A_{\psi_r Z} X, \psi_r X) = 0$. 
Therefore \( g(A_{\psi r}Z(X + Y), \psi r(X + Y)) = 0 \), and hence we obtain
\[
g(A_{\psi r}Z X, \psi r Y) + g(A_{\psi r}Z Y, \psi r X) = 0. \tag{4.5}
\]

Thus using (4.3) and (4.5), we have
\[
g(({\nabla}_Z \psi r)(X), \psi r Y) = g(\bar{\nabla}_Y \psi r X \psi r Y, Z) - g(A_{\psi r}Z X, \psi r Y)
= g(\bar{\nabla}_Y \psi r X, Z) + g(A_{\psi r}Z Y, \psi r X)
= 0. \tag{4.6}
\]

Moreover, it is known that \( D_M \) is a minimal distribution \([2]\), which implies that
\[
(d\psi)(Z, X_1, \ldots, X_{4n}) = 0 \quad \text{for} \quad Z \in D^\perp, \ X_1, \ldots, X_{4n} \in D. \tag{4.7}
\]

Hence
\[
({\nabla}_Z \psi)(X_1, \ldots, X_{4n}) = Z\psi(X_1, \ldots, X_{4n}) - \sum_{a=1}^{4n} \psi(X_1, \ldots, \Pi[Z, X_a], \ldots, X_{4n})
= (d\psi)(Z, X_1, \ldots, X_{4n}) = 0,
\tag{4.8}
\]
which completes the proof. \( \square \)

Now, let \( M \) be a compact totally geodesic quaternion CR-submanifold of a quaternion Kaehler manifold \( \bar{M} \). Let \( N \) be a normal vector field and denote by \( \nu''(N) \) the second normal variation of \( M \) induced by \( N \). Then we have (see [6, Chapter 1]),
\[
\nu''(N) = \int_M \left\{ \left| \nabla^\perp N \right|^2 - \sum_{i=1}^n R(X_i, N, N, X_i) - \left| A_N \right|^2 \right\} dV, \tag{4.9}
\]
where \( N \in T^\perp M, \ dV \) denotes the volume element of \( M \) and \( \{X_i\} \) is an orthonormal frame in \( TM \). Applying the Stokes theorem to the integral of the first term of (4.9), we have
\[
I(N, N) =: \nu''(N) = \int_M g(LN, N) \star 1, \tag{4.10}
\]
where \( L \) is a selfadjoint, strongly elliptic linear differential operator of the second order. The differential operator \( L \) is called the Jacobi operator of \( M \) in \( \bar{M} \) and has discrete eigenvalues \( \lambda_1 < \lambda_2 < \cdots \). We put \( E_\lambda = \{N \in T^\perp M : L(N) = \lambda N\} \). The dimension of the space \( E_\lambda, \dim(E_\lambda) \), is called the index of \( M \) in \( \bar{M} \). For two normal vector fields \( N_1, N_2 \) to a minimal submanifold \( M \) in \( \bar{M} \), their index form is defined by
\[
I(N_1, N_2) = \int_M g(LN_1, N_2) \star 1. \tag{4.11}
\]

It is easy to see that the index form \( I \) is a symmetric bilinear form; \( I : T^\perp M \times T^\perp M \to R \). Now we prove the following theorem.

**Theorem 4.3.** Let \( M \) be a compact \( n \)-dimensional minimal quaternion CR-submanifold of a quaternion Kaehler manifold \( \bar{M} \). If \( M \) has nonpositive holomorphic bisectional curvature, then the index form satisfies
\[
I(N, N) + I(\psi r N, \psi r N) \geq 0 \quad \text{for} \quad N \in \mu. \tag{4.12}
\]
**Proof.** By using the Weingarten equation we have that for all $X,Y \in D^\perp$,

\[
g(\nabla^\perp_X N, \psi_r Y) = g(\nabla_X N, \psi_r Y) = -g(\psi_r \nabla_X N, Y) = -g(\tilde{\nabla}_X \psi_r N, Y) = g(A_{\psi_r N} X, Y) \quad (4.13)
\]

which implies that

\[
\|\nabla^\perp N\|^2 \geq \|A_{\psi_r N}\|^2, \quad \|\nabla^\perp \psi_r N\|^2 \geq \|A_N\|^2 \quad \text{for any } N \in \mu, \quad (4.14)
\]

where $\mu$ is defined in (2.4). Thus by using (4.9), (4.10), (4.13), and (4.14) we get

\[
I(N,N) + I(\psi_r N, \psi_r N) \geq -\int_M \sum_{i=1}^n \{\tilde{R}(N, e_i, e_i, N) + \tilde{R}(\psi_r N, e_i, e_i, \psi_r N)\} \star 1 \quad (4.15)
\]

from which the proof follows, since $M$ has nonpositive holomorphic bisectional curvature.

Finally, we prove a classification theorem for the totally umbilical quaternion CR-submanifolds of a quaternion Kaehler manifold.

**Theorem 4.4.** Let $M$ be a compact totally umbilical quaternion CR-submanifold of a quaternion Kaehler manifold $\bar{M}$. Then

(a) $M$ is a totally geodesic submanifold, or,
(b) $M$ is locally the Riemannian product of a quaternion submanifold and a totally real submanifold, or,
(c) $M$ is a totally real submanifold, or,
(d) the totally real distribution is one dimensional, that is, $\dim D^\perp = 1$,
(e) $\nabla^\perp X H \in \mu$, for $X \in D$.

**Proof.** We take $X, W \in D^\perp$ and using (2.6) with the fact that $\bar{M}$ is a quaternion Kaehler manifold, we have

\[
\psi_r \nabla_X W + g(X,W) \psi_r H = -A_{\psi_r W} X + \nabla_X \psi_r W. \quad (4.16)
\]

Taking inner product with $X$ we get

\[
g(H, \psi_r W) \|X\|^2 = g(X,W) g(H, \psi_r X). \quad (4.17)
\]

Exchanging $X$ and $W$ in (4.17) we have

\[
g(H, \psi_r X) \|W\|^2 = g(X,W) g(H, \psi_r W). \quad (4.18)
\]

This together with (4.17) gives

\[
g(H, \psi_r W) = \frac{g(X,W)^2}{\|X\|^2 \|W\|^2} g(H, \psi_r W). \quad (4.19)
\]
The possible solutions of (4.19) are
(i) $H = 0$,
(ii) $H \perp \psi_r W$,
(iii) $X \parallel W$.

Suppose that condition (i) holds, that is, $H = 0$. This implies that $M$ is totally geodesic which proves (a). Combining (ii) with a result in [1, page 407] we get (b) of the theorem.

Now from (2.7) we have
\[
O = \tilde{R}(IX,JX,KX,N) \\
= \tilde{R}(KX,N,IX,JX) \\
= -\tilde{R}(KX,N,X,KX) \\
= -\tilde{R}(X,KX,KX,N) \\
= -g(\nabla_H N, X)^2 
\]
which implies that
\[
\nabla_H \in \mu \quad \forall X \in D 
\]
proving (e). Next we have
\[
\nabla_X \psi_r H = \psi_r \nabla_X H \quad \text{for } X \in D 
\]
which, by (2.6) gives
\[
\nabla_X \psi_r H = -g(H,H)\psi_r X + \psi_r \nabla_X H. 
\]
Since $\nabla_H \in \mu$, from (4.23) we have $\psi_r X = 0$ for all $X \in D$. Hence $D = \{0\}$ which proves (c). Finally if (iii) is valid then $\dim D^\perp = 1$, which completes the proof.

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