

## A GAUSS TYPE FUNCTIONAL EQUATION

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ABSTRACT. Gauss' functional equation (used in the study of the arithmetic-geometric mean) is generalized by replacing the arithmetic mean and the geometric mean by two arbitrary means.

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**1. Introduction.** By mean we understand a function  $M : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  which satisfies the condition

$$\min(a, b) \leq M(a, b) \leq \max(a, b) \quad \forall a, b > 0. \quad (1.1)$$

The mean is called symmetric if

$$M(a, b) = M(b, a) \quad \forall a, b > 0. \quad (1.2)$$

Usual examples are the power means given by

$$P_n(a, b) = \left( \frac{a^n + b^n}{2} \right)^{1/n} \quad (1.3)$$

for  $n \neq 0$ , while for  $n = 0$  it is the geometric mean

$$P_0(a, b) = G(a, b) = \sqrt{ab}. \quad (1.4)$$

Of course, the arithmetic mean is  $A = P_1$ .

If  $M$  is a mean and  $p : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a strictly monotonous function, the expression

$$M(p)(a, b) = p^{-1}[M(p(a), p(b))] \quad (1.5)$$

defines another mean  $M(p)$  which is called  $M$ -quasi mean (see [1]). For example, the power means are  $A$ -quasi means. More exactly  $P_n = A(e_n)$ , where

$$e_n(x) = x^n \quad \text{for } n \neq 0, \quad e_0(x) = \ln x. \quad (1.6)$$

In what follows, we refer to another famous example of mean. Given two positive numbers  $a$  and  $b$ , we define successively the terms

$$a_{n+1} = A(a_n, b_n), \quad b_{n+1} = G(a_n, b_n), \quad n \geq 0, \quad (1.7)$$

where  $a_0 = a$  and  $b_0 = b$ . It is known (see [1]) that  $(a_n)_{n \geq 0}$  and  $(b_n)_{n \geq 0}$  are convergent to a common limit which is denoted by  $A \otimes G(a, b)$ . It defines the arithmetic-geometric mean of Gauss  $A \otimes G$ .

The following representation formula is also known (see [1])

$$A \otimes G(a, b) = [f(a, b)]^{-1}, \tag{1.8}$$

where

$$f(a, b) = \frac{1}{2 \cdot \pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cdot \cos^2 \theta + b^2 \cdot \sin^2 \theta}}. \tag{1.9}$$

The proof of this formula is based on the fact that the function  $f$  verifies the relation

$$f(A(a, b), G(a, b)) = f(a, b), \tag{1.10}$$

which is called Gauss' functional equation.

These results were generalized as follows. We denote

$$\begin{aligned} r_n(\theta) &= (a^n \cdot \cos^2 \theta + b^n \cdot \sin^2 \theta)^{1/n}, \quad n \neq 0, \\ r_0(\theta) &= \lim_{n \rightarrow 0} r_n(\theta) = a^{\cos^2 \theta} b^{\sin^2 \theta}. \end{aligned} \tag{1.11}$$

If  $p : \mathbb{R}_+ \rightarrow \mathbb{R}$  is a strictly monotonous function, then

$$M_{p,n}(a, b) = p^{-1} \left( \frac{1}{2\pi} \int_0^{2\pi} p(r_n(\theta)) d\theta \right) \tag{1.12}$$

defines a symmetric mean. The arithmetic-geometric mean of Gauss is obtained for  $n = 2$  and  $p(x) = x^{-1}$ . For  $n = -2$  and  $p(x) = x^{-2}$  the mean can be found in [7]. The case  $n = 1$  and  $p = \log$  was studied in [2]. The essential step was done in [4] by the consideration of the definition (1.12) for  $n = 2$  with an arbitrary  $p$ . The values  $n = -1$  and  $n = 1$  were studied in [5, 6]. The general case (of arbitrary  $n$ ) was studied in [8] and continued in [9]. In [8], Gauss' functional equation was also replaced by a more general equation

$$F(P_q(a, b), P_s(a, b)) = F(a, b). \tag{1.13}$$

In this paper, we generalize the mean (1.12) as well as the functional equation (1.13).

**2. An integral mean.** We consider the strictly monotonous functions  $p$  and  $q$ . Using them, we define the functions

$$\begin{aligned} r_q(\theta) &= q^{-1} [q(a) \cdot \cos^2 \theta + q(b) \cdot \sin^2 \theta], \\ f(a, b; p, q) &= \frac{1}{2\pi} \int_0^{2\pi} p[r_q(\theta)] d\theta. \end{aligned} \tag{2.1}$$

It is easy to prove that

$$M_{p,q}(a, b) = p^{-1} [f(a, b; p, q)] \tag{2.2}$$

defines a mean. Choosing  $q = e_n$  we obtain  $M_{p,q} = M_{p,n}$ . We so have generalized the means (1.12). On the other hand, if we let  $p \circ q^{-1} = Q$ , we have  $M_{p,q} = M_{Q,1}(q)$ . Thus,  $M_{p,q}$  is a  $M_{Q,1}$ -quasi mean.

It is thus enough to consider the function

$$f(a, b; p) = \frac{1}{2\pi} \int_0^{2\pi} p(a \cdot \cos^2 \theta + b \cdot \sin^2 \theta) d\theta \tag{2.3}$$

which defines the mean  $M_p = M_{p,1}$  by

$$M_p(a, b) = p^{-1}[f(a, b; p)]. \tag{2.4}$$

In what follows, we assume that the function  $p$  is two times differentiable. From any of the papers [5, 6, 8, 9], we can deduce the following result.

**LEMMA 2.1.** *The function  $f$  defined by (2.3) has the following partial derivatives:*

$$\begin{aligned} f'_a(c, c; p) &= f'_b(c, c; p) = \frac{1}{2} \cdot p'(c), \\ f''_{aa}(c, c; p) &= f''_{bb}(c, c; p) = \frac{3}{8} \cdot p''(c), \\ f''_{ab}(c, c; p) &= \frac{1}{8} \cdot p''(c). \end{aligned} \tag{2.5}$$

**3. The functional equation.** We replace (1.13) by a more general functional equation

$$F(M(a, b), N(a, b)) = F(a, b), \tag{3.1}$$

where  $M$  and  $N$  are two given means. We prove the following result.

**LEMMA 3.1.** *If the function  $f$ , defined by (2.3), verifies the functional equation (3.1), then the function  $p$  is a solution of the differential equation*

$$\begin{aligned} p''(c) \cdot \{ [3 \cdot M'_b(c, c) + N'_b(c, c)] \cdot M'_a(c, c) + [M'_b(c, c) + 3 \cdot N'_b(c, c)] \cdot N'_a(c, c) - 1 \} \\ + 4 \cdot p''(c) \cdot [M''_{ab}(c, c) + N''_{ab}(c, c)] = 0. \end{aligned} \tag{3.2}$$

**PROOF.** Taking in (3.1) the partial derivatives with respect to  $a$ , we obtain

$$F'_a[M(a, b), N(a, b)] \cdot M'_a(a, b) + F'_b[M(a, b), N(a, b)] \cdot N'_a(a, b) = F'_a(a, b). \tag{3.3}$$

Taking again the derivatives with respect to  $b$ , it follows that

$$\begin{aligned} \{ F''_{aa}[M(a, b), N(a, b)] \cdot M'_b(a, b) + F''_{ab}[M(a, b), N(a, b)] \cdot N'_b(a, b) \} \cdot M'_a(a, b) \\ + \{ F''_{ab}[M(a, b), N(a, b)] \cdot M'_b(a, b) + F''_{bb}[M(a, b), N(a, b)] \cdot N'_b(a, b) \} \cdot N'_a(a, b) \\ + F'_a[M(a, b), N(a, b)] \cdot M''_{ab}(a, b) + F'_b[M(a, b), N(a, b)] \cdot N''_{ab}(a, b) = F''_{ab}(a, b). \end{aligned} \tag{3.4}$$

For  $a = b = c$  and the function  $F = f$ , defined by (2.3), we apply Lemma 2.1 and obtain (3.2). □

**CONSEQUENCE 3.2.** *If the function  $f$ , defined by (2.3), verifies the functional equation (3.1), where the means  $M$  and  $N$  are symmetric, the function  $p$  is a solution of the differential equation*

$$p''(c) + 4 \cdot p'(c) \cdot [M''_{ab}(c, c) + N''_{ab}(c, c)] = 0. \tag{3.5}$$

**PROOF.** As the means are symmetric, their partial derivatives of the first order are equal to  $1/2$  (see [3]), thus (3.2) becomes (3.5). □

**CONSEQUENCE 3.3.** If the function  $f$ , defined by (2.3), verifies the functional equation (1.13), then the function  $p$  is given by

$$p(c) = C \cdot c^{r+s-1} + D, \tag{3.6}$$

where  $C$  and  $D$  are arbitrary constants.

**PROOF.** We have in (3.5),  $M = P_r$  and  $N = P_s$ . Thus

$$M''_{ab}(c, c) = \frac{1-r}{4 \cdot c}, \quad N''_{ab}(c, c) = \frac{1-s}{4 \cdot c}. \tag{3.7}$$

Replacing them in (3.5), we obtain the differential equation

$$p''(c) + \frac{2-r-s}{c} \cdot p'(c) = 0 \tag{3.8}$$

with the solution given above. □

**REMARK 3.4.** This last result was obtained in [8]. As it is shown in [9], the condition is also sufficient for  $r = -s = 1$ .

**REMARK 3.5.** Equation (3.1) can be further generalized at

$$F(g(M(a, b)), g(N(a, b))) = h(F(a, b)), \tag{3.9}$$

where  $g$  and  $h$  are two given functions. We have in view the following result given in [2]. The function  $f$ , defined by (2.3), verifies the relation

$$f(A^2(a, b), G^2(a, b); \log) = 2 \cdot f(a, b; \log). \tag{3.10}$$

**4. Special means.** A problem which is studied for the integral means defined in [4, 5, 6, 8, 9] is that of the determination of the cases in which the mean reduces at a given one, usually a power mean. Similar results can be given also in more general circumstances. We prove the following lemma.

**LEMMA 4.1.** *If for a given mean  $N$ , we have  $M_p = N$ , then the function  $p$  verifies the equation*

$$p''(c) \cdot [8 \cdot N'_a(c, c) \cdot N'_b(c, c) - 1] + 8 \cdot p'(c) \cdot N''_{ab}(c, c) = 0. \tag{4.1}$$

**PROOF.** In the given hypotheses, we have

$$f(a, b; p) = p[N(a, b)]. \tag{4.2}$$

Taking the partial derivatives with respect to  $a$ , we have

$$f'_a(a, b; p) = p'[N(a, b)] \cdot N'_a(a, b). \tag{4.3}$$

Then we take the derivative with respect to  $b$ , we obtain

$$f''_{ab}(a, b; p) = p''[N(a, b)] \cdot N'_a(a, b) \cdot N'_b(a, b) + p'[N(a, b)] \cdot N''_{ab}(a, b). \tag{4.4}$$

For  $a = b = c$ , Lemma 2.1 gives (4.1). □

**CONSEQUENCE 4.2.** If we have  $M_p = N$ , with the symmetric mean  $N$ , then the function  $p$  verifies the equation

$$p''(c) + 8 \cdot p'(c) \cdot N_{ab}''(c, c) = 0. \quad (4.5)$$

**CONSEQUENCE 4.3.** If we have  $M_p = P_r$ , then the function  $p$  is given by

$$p(c) = C \cdot c^{2r-1} + D, \quad (4.6)$$

where  $C$  and  $D$  are arbitrary constants.

**REMARK 4.4.** In [9], it is shown that the above condition is also sufficient for  $r = 0$ ,  $1/2$ , and  $1$ .

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