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PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The periodic boundary value problems of a class of nonlinear differential equations are investigated.

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1. Introduction. Let us consider the following nonlinear differential equation

$$(\bar{a}(t)\bar{\varphi}_p(x'(t)))' + f(t,x(t)) = 0, \tag{1.1}$$

where $' = d/dt$, $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous, 2π-periodic in t and $f(t,\cdot) \in \mathbb{C}^1(\mathbb{R}^n,\mathbb{R}^n)$, $\bar{a}(t)\bar{\varphi}_p(x) =: \mathrm{col}(a_1(t)\varphi_p(x_1),\dots,a_n(t)\varphi_p(x_n))$, $a_k(t)$ is 2π-periodic and $a_k(t) \in \mathbb{C}^1(\mathbb{R},(0,\infty))$, $\varphi_p : \mathbb{R}\to\mathbb{R}$ be defined by $\varphi_p(s) = |s|^{p-2}s$, with $p > 1$ fixed, $f(t,x) = \mathrm{col}(f_1(t,x),\dots,f_n(t,x))$.

When $p = 2$, $a_k(t) \equiv 1$, $k = 1,2,\dots,n$. Equation (1.1) is of the form

$$x''(t) + f(t,x) = 0. \tag{1.2}$$

Amaral and Pera [1] and recently Li [5] proved the existence and uniqueness results of (1.2) under the following assumptions:

(L) There exist two constant symmetric $n\times n$ matrices A_0 and B_0 with eigenvalues N_k^2 and $(N_k+1)^2$, $(k = 1,2,\dots,n)$, respectively, $N_k \ge 0$ is an integer for each k, $D_2f(t,x)$ is symmetric and there exist two time-dependent continuous symmetric $n\times n$ matrices $A(t)$ and $B(t)$ such that $A_o \le A(t) \le D_2f(t,x) \le B(t) \le B_0$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$. Furthermore, $N_k^2 < \lambda_k(t) \le \mu_k(t) < (N_k+1)^2$ on a subset of $[0,2\pi]$ of positive measure, where $\lambda_k(t)$ and $\mu_k(t)$ are the eigenvalues of $A(t)$ and $B(t)$, respectively.

Inspired by the work of Li [5], we give sufficient conditions for the existence and uniqueness of the 2π-periodic solution of (1.1) by using the initial value problem method and homeomorphism of $\mathbb{R}^n$ to $\mathbb{R}^n$.

2. Initial value problem and eigenvalues problem. Throughout this paper, we denote the interval $[0,2\pi]$ and M_n denotes the set of all complex $n\times n$ matrices. We also assume the solutions of (1.1) exist on I for any initial value $(x(0),x'(0)) \in \mathbb{R}^{2n}$.

Let us consider the initial value problem

$$u'(t) = g(t,u(t)), \qquad u(0) = u_0. \tag{2.1}$$

Lemma 2.1 [3, 4]. *Assume that* $g \in \mathbb{C}(I \times \mathbb{R}^n, \mathbb{R}^n)$ *and possesses continuous partial derivatives* $\partial g/\partial u$ *on* $I \times \mathbb{R}^n$. *Let the solution* $u_0(t) = u(t, t_0, u_0)$ *of (2.1) exist for* $t \in I$ *and let*

$$H(t, t_0, u_0) = \frac{\partial g(t, u(t, t_0, u_0))}{\partial u}. \tag{2.2}$$

Then

$$\phi(t, t_0, u_0) = \frac{\partial u(t, t_0, u_0)}{\partial u_0} \tag{2.3}$$

exists and is the solution of

$$V' = H(t, t_0, u_0)V \tag{2.4}$$

such that $\phi(t_0, t_0, u_0)$ *is the unit matrix.*

Lemma 2.2 [2]. *Suppose* $A \in M_n$. *Then* λ *is an eigenvalue of the matrix* A *if and only if* $\exp\lambda$ *is an eigenvalue of the matrix* $\exp A$.

Lemma 2.3 [2]. *If* $A \in M_n$ *and there exists* $\delta > 0$ *such that* $|\lambda| > \delta$ *for all eigenvalues* λ *of* A, *then* $\|A^{-1}\| \le \delta^{-n}\|A\|^{n-1}$, *where* $\|A\| = \max \lambda^{1/2}(A^*A)$ [A^* *denotes the adjoint of* A, *i.e., if* $A = (a_{ij})$, $A^* = (\bar{a}_{ji})$].

Lemma 2.4 [7]. *If* $A \ge B \ge 0$, *and* A *and* B *are two real symmetric* $n \times n$ *matrices, where* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ *and* $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ *are eigenvalues of* A *and* B, *respectively, then* $\lambda_k \ge \mu_k$, *for* $k = 1, 2, \ldots, n$.

Lemma 2.5 [6]. *Assume that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is continuously differentiable on* $\mathbb{R}^n$ *and* $\|[F'(x)]^{-1}\| \le M < +\infty$ *for all* $x \in \mathbb{R}^n$. *Then* F *is a homeomorphism of* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

Lemma 2.6. *Assume* A, B *are* $n \times n$ *matrices, then the eigenvalues of the* $2n \times 2n$ *matrix*

$$\begin{pmatrix} 0 & \bar{A} \\ -\bar{B} & 0 \end{pmatrix} \tag{2.5}$$

are the roots of $\det(\lambda^2 I_n + \overline{BA}) = 0$.

Proof. From the following matrices equality

$$\begin{pmatrix} \lambda I_n & 0 \\ -\bar{B} & \lambda I_n \end{pmatrix} \begin{pmatrix} \lambda I_n & -\bar{A} \\ \bar{B} & \lambda I_n \end{pmatrix} = \begin{pmatrix} \lambda^2 I_n & -\lambda\bar{A} \\ 0 & \lambda^2 I_n + \overline{BA} \end{pmatrix} \tag{2.6}$$

we obtain the result of Lemma 2.6 immediately. □

3. Main results. Rewrite (1.1) as follows

$$x' = \varphi_q(b(t)y), \qquad y' = -f(t, x), \tag{3.1}$$

where $b(t)y =: \mathrm{col}(b_1(t)y_1(t), \ldots, b_n(t)y_n(t))$, $b_k(t) = a_k^{-1}(t)$, and $(1/p) + (1/q) = 1$, $(q = p/(p-1))$, $y_k(t) = a_k(t)\varphi_p(x_k')$, hence $x_k'(t) = \varphi_q(b_k(t)y_k(t))$, $k = 1, 2, \ldots, n$.

Let $u = \mathrm{col}(x,y) \in \mathbb{R}^{2n}$, $g(t,u) = \mathrm{col}(\varphi_q(b(t)y), -f(t,x)) \in \mathbb{R}^{2n}$, $v = \mathrm{col}(\alpha,\beta) = \mathrm{col}(x(0),y(0)) = \mathrm{col}(x(0),a(0)\varphi_p(x'(0))) \in \mathbb{R}^{2n}$, then (3.1) is of the form

$$u'(t) = g(t,u(t)), \qquad u(0) = v. \tag{3.2}$$

Consider the variation equation of (3.2) with respect to u

$$\xi' = \frac{\partial g(t,u)}{\partial u}\xi, \tag{3.3}$$

where

$$\frac{\partial g(t,u)}{\partial u} = \begin{pmatrix} 0 & \dfrac{\partial \varphi_q(b(t)y)}{\partial y} \\ \dfrac{-\partial f(t,x)}{\partial x} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & A \\ -B & 0 \end{pmatrix} \tag{3.4}$$

with

$$\begin{aligned} A &= (q-1)\,\mathrm{diag}\left(b_1(t)|y_1(t)|^{q-2},\dots,b_n(t)|y_n(t)|^{q-2}\right), \\ B &= \frac{\partial f(t,x(t))}{\partial x} = \nabla f(t,x(t)). \end{aligned} \tag{3.5}$$

Let

$$Z(t) = \exp\int_0^t \frac{\partial g(s,u(s,v))}{\partial u}\,ds. \tag{3.6}$$

Then $Z(t)$ is a fundamental solution matrix of (3.3) and $Z(0) = I_{2n}$. Meanwhile, by Lemma 2.1 we know that

$$\frac{\partial u}{\partial v} = \begin{pmatrix} \dfrac{\partial x(t,v)}{\partial \alpha} & \dfrac{\partial x(t,v)}{\partial \beta} \\ \dfrac{\partial y(t,v)}{\partial \alpha} & \dfrac{\partial y(t,v)}{\partial \beta} \end{pmatrix} \tag{3.7}$$

is also a fundamental solution matrix of (3.3). Therefore

$$Z(t) = \frac{\partial u(t,v)}{\partial v}, \quad t \in [0,2\pi]. \tag{3.8}$$

Define: $h,H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $h(v) = \mathrm{col}(x(2\pi,v),y(2\pi,v))$,

$$H(v) = v - h(v). \tag{3.9}$$

By Lemma 2.1, $h(v)$ is $\mathbb{C}^1$-differentiable and so is $H(v)$. Therefore, solving periodic solution of (1.1) is equivalent to finding the fixed points of $h(v)$ or the zero points of $H(v)$. From (3.8) and (3.9)

$$\begin{aligned} H'(v) &= I_{2n} - h'(v) = I_{2n} - \frac{\partial u(2\pi,v)}{\partial v} = I_{2n} - Z(2\pi) \\ &= I_{2n} - \exp\int_0^{2\pi} \frac{\partial g(t,u(t,v))}{\partial u}\,dt. \end{aligned} \tag{3.10}$$

Theorem 3.1. *Let* $\bar{A}_v = (q-1)\operatorname{diag}(\int_0^{2\pi} b_1(t)|\beta_1 - \int_0^t f_1(\tau,x(\tau))\,d\tau|^{q-2}dt,\dots,\int_0^{2\pi} b_n(t)|\beta_n - \int_0^t f_n(x,x(\tau))d\tau|^{q-2}\,dt)$, $\bar{B}_v = \int_0^{2\pi}\nabla f(t,x(t))\,dt$, *where* $x(t)$ *is any solution of (1.1) satisfying initial conditions* $(x(0),y(0)) = v = (\alpha,\beta)\in\mathbb{R}^{2n}$. *If there exist* $v\in\mathbb{R}^{2n}$ *and integers* $N_k \ge 0$, $k=1,2,\dots,n$, *such that the matrix* $\bar{B}_v\bar{A}_v$ *is similar to a diagonal matrix* $C_v = \operatorname{diag}(\lambda_1,\dots,\lambda_n)$ *with* $(2\pi N_k)^2 < \lambda_k < [2\pi(N_k+1)]^2$, $k = 1,2,\dots,n$. *Then (1.1) has a unique* 2π*-periodic solution* $x(t)$ *satisfying the initial condition* $(x(0),y(0)) = v$.

Proof. By Lemma 2.5, we need only to show that $H'(V)$ is invertible and that there exists a constant $M>0$ such that $\|[H'(v)]^{-1}\|\le M$.

In fact, from (3.1) and (3.5), $A = (q-1)\operatorname{diag}(b_1(t)|y_1(t)|^{q-2},\dots,b_n(t)|y_n(t)|^{q-2})$, since

$$y_k(t) = y_k(0) - \int_0^t f_k(\tau,x(\tau))\,d\tau = \beta_k - \int_0^t f_k(\tau,x(\tau))\,d\tau,\quad k=1,2,\dots,n \tag{3.11}$$

we have

$$\begin{aligned}\bar{A}_v = \int_0^{2\pi} A\,dt = (q-1)\operatorname{diag}\Bigg(&\int_0^{2\pi} b_1(t)\left|\beta_1 - \int_0^t f_1(\tau,x(\tau))\,d\tau\right|^{q-2}dt,\dots,\\ &\times\int_0^{2\pi} b_n(t)\left|\beta_n - \int_0^t f_n(\tau,x(\tau))\,d\tau\right|^{q-2}dt\Bigg).\end{aligned} \tag{3.12}$$

From Lemma 2.6, the eigenvalues of the matrix $\begin{pmatrix}0 & \bar{A}_v\\ -\bar{B}_v & 0\end{pmatrix}$ are $\pm\sqrt{\lambda_1}i,\pm\sqrt{\lambda_2}i,\dots,\pm\sqrt{\lambda_n}i$. By (3.5), (3.10), and Lemma 2.2 the eigenvalues of $H'(v)$ are

$$\mu_k = 1-\exp\left(\pm\sqrt{\lambda_k}i\right) = 1-\cos\sqrt{\lambda_k}\mp i\sin\sqrt{\lambda_k},\quad k=1,2,\dots,n. \tag{3.13}$$

From the assumption of λ_k and

$$|\mu_k| = \sqrt{2-2\cos\sqrt{\lambda_k}} = 2\left|\sin\frac{\sqrt{\lambda_k}}{2}\right| \tag{3.14}$$

it follows that

$$|\mu_k| \ge 2\min_{1\le k\le n}\left(\left|\sin\frac{\sqrt{\lambda_k}}{2}\right|\right) > 0 \tag{3.15}$$

because $N_k\pi < \sqrt{\lambda_k}/2 < (N_k+1)\pi$, $k=1,2,\dots,n$,

$$\|[H'(v)]^{-1}\| \le \frac{1+\exp\left(4\max_{1\le k\le n}(N_k+1)^2\pi^2\right)}{\left(2\min\{|\sin\sqrt{\lambda_k}/2|\}\right)^n} = M. \tag{3.16}$$

Now, from Lemma 2.5, $H'(v)$ is invertible, since by Lemma 2.3, $H'(v)$ is homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$, hence there exists a unique $v_0\in\mathbb{R}^n$ such that $H(v_0)=0$, that is, $h(v_0)=v_0$. Theorem 3.1 is proved. □

Corollary 3.2. *Let* $p=2$, $a_k(t)\equiv 1$, $k=1,2,\dots,n$ *in (1.1), and suppose* (L) *holds, then (1.1) has a unique* 2π*-periodic solution.*

Proof. In this case, $q = 2$, hence $\bar{A} = 2\pi I_n$, then eigenvalues of $\begin{pmatrix} 0 & \bar{A} \\ -\bar{B} & 0 \end{pmatrix}$ are

$$\pm\sqrt{2\pi\lambda_k}\,i, \quad k = 1,2,\ldots,n \tag{3.17}$$

with

$$2\pi N_k^2 < \lambda_k < 2\pi(N_k+1)^2, \tag{3.18}$$

therefore

$$\|[H'(v)]^{-1}\| \le \frac{1+\exp\left(2\max_{1\le k\le N}(N_k+1)^2\pi\right)}{\left(2\min_{1\le k\le N}\{\sin|\sqrt{2\pi a_k}/2|,\sin|\sqrt{2\pi b_k}/2|\}\right)^2} = M, \tag{3.19}$$

where

$$2\pi N_k^2 < a_k = \int_0^{2\pi}\lambda_k(t)\,dt \le \int_0^{2\pi}\mu_k(t)\,dt = b_k < 2\pi(N_k+1)^2. \tag{3.20}$$

From Lemma 2.3 again, (1.1) has a unique 2π-periodic solution. Corollary 3.2 is proved. □

Remark 3.3. Corollary 3.2 is the result of [1] and [5].

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