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A THEOREM OF MEIR-KEELER TYPE REVISITED

Y. J. CHO, P. P. MURTHY, and G. JUNGCK

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Abstract. In 1993, the authors presented a fixed point theorem of Meir-Keeler type. The proposed proof of a lemma—on which the said theorem depends on—is invalid. In this note, we alter the statement of this lemma and give a valid proof thereof, so that the main result of the previous paper is still true.

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In 1993, we introduced the concept of compatible maps of type (A) and "proved" the following theorem.

Theorem 1 [1, Theorem 3.2]. *Let A,B,S and T be mappings of a complete metric space (X,d). Suppose that the pair $\{A,B\}$ is a generalized (ϵ,δ)-$\{S,T\}$-contraction with δ lower semi-continuous. If the following conditions are satisfied:*
(i) *one of A,B,S or T is continuous, and*
(ii) *the pairs A,S and B,T are compatible of type (A) on X,*
then A,B,S and T have a unique common fixed point in X.

The purpose of this note is to ensure that the above is indeed true. This is necessary since the proof of Theorem 1 relies on Lemma 3.1 in [1]. However, the proof of part (1) of this lemma is faulty and the proof of part (2) is not "tight." In the following, we provide a thorough and complete proof of Lemma 4 below which is a "reshuffled and revamped" version of Lemma 3.1 in [1]. This accomplishes our mission, since the proof of Theorem 1 is valid if the lemma is true.

The proof of part (3) of Lemma 4 below is much like the proof of Lemma 3.1(c) in [2] with minor initial modifications. We include all the proof of part (3) for ease of reading and completeness sake. We need the following definitions given in [1].

Definition 2 [2]. Let A,B,S and T be mappings of a metric space (X,d) into itself such that $A(X)\subset T(X)$ and $B(X)\subset S(X)$. For $x_0\in X$, any sequence $\{y_n\}$ defined by

$$\begin{aligned} y_{2n-1} &= Tx_{2n-1} = Ax_{2n-2}, \\ y_{2n} &= Sx_{2n} = Bx_{2n-1} \end{aligned} \tag{1}$$

for $n\in\mathbb{N}$ (the set of positive integers) is called an $\{S,T\}$-iteration of x_0 under A and B.

The following definition was given in [1], but erroneously required that $\delta(\epsilon)<\epsilon$.

Definition 3. Let A, B, S and T be mappings of a metric space (X,d) into itself. The pair $\{A,B\}$ is called a *generalized* (ϵ,δ)-$\{S,T\}$*-contraction* if

$$A(X) \subset T(X), \qquad B(X) \subset S(X) \tag{2}$$

and there exists a function $\delta : (0,\infty) \longrightarrow (0,\infty)$ such that, for any $\epsilon > 0, \delta(\epsilon) > \epsilon$, and

$$\begin{aligned}\epsilon \le M(x,y) = \max\Big\{&d(Sx,Ty), d(Sx,Ax), d(Ty,By),\\ &\frac{1}{2}\big(d(Sx,By)+d(Ty,Ax)\big)\Big\} < \delta(\epsilon) \Rightarrow d(Ax,By) < \epsilon.\end{aligned} \tag{3}$$

Now we state and prove a modified version of the lemma in question.

Lemma 4. *Let A, B, S and T be mappings of a metric space (X,d) into itself and let the pair $\{A,B\}$ be a generalized (ϵ,δ)-$\{S,T\}$-contraction. If $x_0 \in X$ and $\{y_n\}$ is an $\{S,T\}$ iteration of x_0 under A and B, we have the following:*

(1) $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

(2) For every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that, whenever $p, q \ge n_1$ and of opposite parity,

$$\epsilon \le d(y_p, y_q) < \epsilon + r \Longrightarrow d(y_{p+1}, y_{q+1}) < \epsilon, \tag{4}$$

where $r = \min\{\epsilon/2, (\delta(\epsilon)-\epsilon)/2\}$.

(3) The sequence $\{y_n\}$ is a Cauchy sequence in X.

Proof. To prove part (1), first note that, by (3),

$$\begin{aligned}&d(Ax,By) = 0, \quad \text{if } M(x,y) = 0,\\ &d(Ax,By) < M(x,y), \quad \text{otherwise.}\end{aligned} \tag{5}$$

Thus, $d(Ax,By) \le M(x,y)$ for $x, y \in X$. Therefore, if $x_0 \in X$, (1) and (3) imply that

$$\begin{aligned}d(y_{2n}, y_{2n+1}) &= d(Bx_{2n-1}, Ax_{2n})\\ &= d(Ax_{2n}, Bx_{2n-1}) \le M(x_{2n}, x_{2n-1})\\ &= \max\Big\{d(Sx_{2n}, Tx_{2n-1}), d(Sx_{2n}, Ax_{2n}), d(Tx_{2n-1}, Bx_{2n-1}),\\ &\qquad \frac{1}{2}\big(d(Sx_{2n}, Bx_{2n-1}) + d(Tx_{2n-1}, Ax_{2n})\big)\Big\}\\ &= \max\Big\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}),\\ &\qquad \frac{1}{2}\big(d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\big)\Big\}\\ &\le \max\Big\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}),\\ &\qquad \frac{1}{2}\big(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\big)\Big\}\\ &\le \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}.\end{aligned} \tag{6}$$

Now if $M(x_{2n}, x_{2n-1}) = 0$, by the above, we know

$$d(y_{2n}, y_{2n-1}) = d(y_{2n}, y_{2n+1}) = 0. \tag{7}$$

But if $M(x_{2n},x_{2n-1}) > 0$, (5) and the above imply that

$$d(y_{2n},y_{2n+1}) < M(x_{2n},x_{2n-1}) \le \max\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1})\}, \tag{8}$$

i.e.,

$$d(y_{2n},y_{2n+1}) < d(y_{2n},y_{2n-1}). \tag{9}$$

Thus, in any event, we have

$$d(y_{2n},y_{2n+1}) \le M(x_{2n},x_{2n-1}) \le d(y_{2n},y_{2n-1}) \tag{10}$$

for $n \in X$. Similarly,

$$d(y_{2n+1},y_{2n+2}) \le M(x_{2n+1},x_{2n}) \le d(y_{2n},y_{2n+1}) \tag{11}$$

for $n \in X$. Thus $s = \{d(y_k,y_{k+1})\}$ is nonincreasing and is bounded below by 0. Hence, s converges to $t \in [0,\infty)$, the greatest lower bound of s. If $t = 0$, we are done. So, suppose that $t > 0$. Since s converges in a nonincreasing manner to t, (10) yields $m \in \mathbb{N}$ such that

$$t \le M(x_{2m},x_{2m-1}) < \delta(t). \tag{12}$$

But then (3) implies that

$$d(Ax_{2m},Bx_{2m-1}) = d(y_{2m+1},y_{2m}) < t, \tag{13}$$

which contradicts the fact that t is the greatest lower bound of s. Thus part (1) is true.

Now we prove part (2). Let $\epsilon > 0$. Part (1) permits us to choose $n_1 \in \mathbb{N}$ such that

$$d(y_n,y_{n+1}) < \frac{r}{2} \quad \text{for } n \ge n_1, \tag{14}$$

where $r = \min\{\epsilon/2,(\delta(\epsilon)-\epsilon)/2\}$. Let $p,q \in \mathbb{N}$ such that $p,q \ge n_1$, where $p = 2n$ and $q = 2m-1$. Suppose that

$$\epsilon \le d(y_p,y_q) = d(y_{2n},y_{2m-1}) < \epsilon + r. \tag{15}$$

Keeping (1), (3), (14) and (15) in mind, we can write the following:

$$\begin{aligned}
\epsilon \le d(y_p,y_q) &= d(Sx_{2n},Tx_{2m-1}) \le M(x_{2n},x_{2m-1})\\
&= \max\Big\{d(Sx_{2n},Tx_{2m-1}),d(Sx_{2n},Ax_{2n}),d(Tx_{2m-1},Bx_{2m-1}),\\
&\qquad \frac{1}{2}(d(Sx_{2n},Bx_{2m-1})+d(Tx_{2m-1},Ax_{2n}))\Big\}\\
&= \max\Big\{d(y_p,y_q),d(y_p,y_{p+1}),d(y_q,y_{q+1}),\\
&\qquad \frac{1}{2}(d(y_p,y_{q+1}),d(y_q,y_{p+1}))\Big\}\\
&= \max\Big\{d(y_p,y_q),\frac{1}{2}(d(y_p,y_{q+1})+d(y_q,y_{p+1}))\Big\}\\
&\le \max\Big\{d(y_p,y_q),\frac{1}{2}(2d(y_p,y_q)+d(y_q,y_{q+1})+d(y_p,y_{p+1}))\Big\}\\
&\le d(y_p,y_q)+\frac{r}{2} < \epsilon + \frac{3r}{2} < \epsilon + 2r \le \epsilon + (\delta(\epsilon)-\epsilon) = \delta(\epsilon).
\end{aligned} \tag{16}$$

Thus we have $\epsilon \le M(x_{2n}, x_{2m-1}) < \delta(\epsilon)$, and so (3) implies that

$$d(y_{p+1}, y_{q+1}) = d(Ax_{2n}, bx_{2m-1}) < \epsilon, \tag{17}$$

as desired.

To prove part (3), let $\alpha = 2\epsilon > 0$ and let $r = \min\{\epsilon/2, (\delta(\epsilon)-\epsilon)/2\}$. Part (2) of the lemma yields $n_1 \in \mathbb{N}$ such that, whenever $p, q \in \mathbb{N}$ and $p, q > n_1$, then

$$d(y_{p+1}, y_{q+1}) < \epsilon \text{ if } \epsilon \le d(y_p, y_q) < \epsilon + r \text{ and } p, q \text{ are of opposite parity.} \tag{18}$$

And part (1) of the lemma permits us to choose $n_0 \in \mathbb{N}$ such that $n_0 > n_1$ and

$$d(y_m, y_{m+1}) < \frac{r}{6} \tag{19}$$

for $m \ge n_0$. Now we let $q > p \ge n_0$—so that both (18) and (19) hold—and show that $d(y_p, y_q) < \alpha$, thereby proving that $\{y_n\}$ is a Cauchy sequence in X. So suppose that

$$d(y_p, y_q) \ge \alpha = 2\epsilon. \tag{20}$$

To show that (20) produces a contradiction, we first want to choose an $m > p$ such that

$$\epsilon + \frac{r}{3} < d(y_p, y_m) < \epsilon + r \text{ with } p \text{ and } m \text{ of opposite parity.} \tag{21}$$

To this end, let k be the smallest integer greater than p such that $d(y_p, y_k) > \epsilon + (r/2)$. The integer k exists by (20) since $r < \epsilon$. Moreover, we have

$$d(y_p, y_k) < \epsilon + \frac{2r}{3}. \tag{22}$$

For otherwise, $\epsilon + 2r/3 \le d(y_p, y_{k-1}) + d(y_{k-1}, y_k) < d(y_p, y_{k-1}) + r/6$, since $k-1 \ge p \ge n_0 > n_1$, and therefore

$$\epsilon + \frac{r}{2} < d(y_p, y_{k-1}). \tag{23}$$

Since $k-1 \ge p$, (23) implies that $k-1 > p$. But then (23) contradicts the choice of k. We thus have

$$\epsilon + \frac{r}{2} < d(y_p, y_k) < \epsilon + \frac{2r}{3}. \tag{24}$$

So, if p and k are of opposite parity, we can let $m = k$ in (24) to obtain (21). If p and k are of like parity, p and $k+1$ are opposite parity. Since $d(y_k, y_{k+1}) < r/6$ by (19), the triangle inequality and (24) imply that

$$\epsilon + \frac{r}{3} < d(y_p, y_{k+1}) < \epsilon + \frac{5r}{6}. \tag{25}$$

In this instance, we let $m = k+1$. In any event, by (24) and (25), we can choose m such that m and p are of opposite parity and (21) holds. But then, since $p, m \ge n_0$, (19) and (21) imply that

$$\epsilon + \frac{r}{3} < d(y_p, y_m) \le d(y_p, y_{p+1}) + d(y_{p+1}, y_{m+1}) + d(y_{m+1}, y_m). \tag{26}$$

Therefore, by (21) and (18), we have

$$\epsilon + \frac{r}{3} < \frac{r}{3} + d(y_{p+1}, y_{m+1}) < \frac{r}{3} + \epsilon. \tag{27}$$

This is the anticipated contradiction. This completes the proof. □

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Cho: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: yicho@nongae.gsnu.ac.kr

Murthy: Department of Mathematics, Arignar Anna Govt. Arts College, Karaikal (Pondicherry V. T.) 609605, India

Jungck: Department of Mathematics, Bradley University, Peoria, IL 61625, USA