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ON 3-TOPOLOGICAL VERSION OF Θ-REGULARITY

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ABSTRACT. We modify the concept of θ-regularity for spaces with 2 and 3 topologies. The new, more general property is fully preserved by sums and products. Using some bitopological reductions of this property, Michael's theorem for several variants of bitopological paracompactness is proved.

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1. Preliminaries. The term *space* (X,τ,σ,ρ) is referred as a set X with three, generally nonidentical topologies τ,σ, and ρ. We say that $x \in X$ is a (σ,ρ)-*θ-cluster point* of a filter base Φ in X if for every $V \in \sigma$ such that $x \in V$ and every $F \in \Phi$ the intersection $F \cap \mathrm{cl}_\rho V$ is nonempty. If, for every $V \in \sigma$ with $x \in V$, there is some $F \in \Phi$ with $F \subseteq \mathrm{cl}_\rho V$, we say that $\Phi(\sigma,\rho)$-*θ-converges* to x. Then x is called a (σ,ρ)-*θ-limit* of Φ. If Φ converges or has a cluster point with respect to the topology τ, we say that Φ τ-*converges* or has a τ-*cluster point*.

A family is called σ-*locally finite* if it consists of countably many locally finite subfamilies. (This notion has nothing common with the topology also denoted by σ.) For a family $\Phi \subseteq 2^X$, we denote by Φ^F the family of all finite unions of members of Φ. A family Φ is called *directed* if Φ^F is a refinement of Φ.

We say that the space (X,τ,σ,ρ) is $(\tau-\sigma)$ *(semi-) paracompact with respect to* ρ if every τ-open cover of X has a σ-open refinement which is (σ-) locally finite with respect to the topology ρ.

The bitopological space (X,τ,σ) is called RR-*pairwise (semi-) paracompact* if the space is $(\tau-\tau)$ (semi-) paracompact with respect to σ and $(\sigma-\sigma)$ (semi-) paracompact with respect to τ. We say that (X,τ,σ) is *FHP-pairwise (semi-) paracompact* if the space is $(\tau-\sigma)$ (semi-) paracompact with respect to σ and $(\sigma-\tau)$ (semi-) paracompact with respect to τ. Finally, (X,τ,σ) is said to be *δ-pairwise (semi-) paracompact* if the space is $(\tau-(\tau\vee\sigma))$ (semi-) paracompact with respect to $\tau\vee\sigma$ and $(\sigma-(\tau\vee\sigma))$ (semi-) paracompact with respect to $\tau\vee\sigma$ (see [7]).

Recall that the topological space (X,τ) is called *(countably)* θ-regular [2] if every (countable) filter base in (X,τ) with a θ-cluster point has a cluster point.

2. Main results

THEOREM 2.1. *Let τ, σ, ρ be topologies on X. The following statements are equivalent:*

(i) *For every (countable) τ-open cover Ω of X and each $x \in X$ there is a σ-open* *neighborhood U of x such that $\mathrm{cl}_\rho U$ can be covered by a finite subfamily of Ω.*

(ii) *Every (countable) τ-closed filter base Φ with a (σ,ρ)-θ-cluster point has a τ-cluster point.*

(iii) *Every (countable) filter base Φ with a (σ,ρ)-θ-cluster point has a τ-cluster point.*

(iv) *For every (countable) filter base Φ in X with no τ-cluster point and every $x \in X$ there are $U \in \sigma$, $V \in \rho$, and $F \in \Phi$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$.*

Proof. Suppose (i). Let Φ be a (countable) filter base in X with no τ-cluster point. Then $\Omega = \{X \smallsetminus \mathrm{cl}_\tau F \mid F \in \Phi\}$ is a (countable) τ-open directed cover of X. Let $x \in X$. By (i) there is $U \in \sigma$ with $x \in U$ and $\mathrm{cl}_\rho U \subseteq X \smallsetminus \mathrm{cl}_\tau F$ for some $F \in \Phi$. Denote $V = X \smallsetminus \mathrm{cl}_\rho U$. Then $x \in U$, $F \subseteq V \in \rho$ and $U \cap V = \emptyset$. It follows (iv).

The implications (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii) are clear. Suppose (ii). Take any (countable) τ-open cover Ω of X. Then $\Phi = \{X \smallsetminus V \mid V \in \Omega^F\}$ is a τ-closed filter base in X with no τ-cluster point. Let $x \in X$. It follows from (ii) that x is not a (σ,ρ)-θ-cluster point of Φ, so there are some $U \in \sigma$ and $V \in \Omega^F$ such that $x \in U$ and $(X \smallsetminus V) \cap \mathrm{cl}_\rho U = \emptyset$. Then $\mathrm{cl}_\rho U \subseteq V$, which implies (i). □

Definition 2.2. Let τ, σ, ρ be topologies on X. Then (X,τ,σ,ρ) is said to be *(countably) (τ,σ,ρ)-θ-regular*, if X satisfies any of the conditions (i)–(iv) of Theorem 2.1.

Note that for $\tau = \sigma = \rho$ we obtain the notion of (countably) θ-regular space. Omitting the condition of countability, we get further criteria of (τ,σ,ρ)-θ-regularity.

Theorem 2.3. *Let τ, σ, ρ be topologies on X. The following statements are equivalent:*

(i) *X is (τ,σ,ρ)-θ-regular.*

(ii) *Every (σ,ρ)-θ-convergent filter base Φ has a τ-cluster point.*

(iii) *Every (σ,ρ)-θ-convergent ultrafilter in X is τ-convergent.*

Proof. The implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) are clear. Conversely, suppose (iii) and take a filter base Φ in X with a (σ,ρ)-θ-cluster point $x \in X$. Let ζ be a σ-open local base of x. Then the family $\Phi' = \{F \cap \mathrm{cl}_\rho V \mid F \in \Phi, V \in \zeta\}$ is a filter base finer than Φ and (σ,ρ)-θ-converging to x. Denote by Γ an ultrafilter finer than Φ'. Then $\Phi' \subseteq \Gamma$ and hence Γ also (σ,ρ)-θ-converges to x. By (iii), Γ is τ-convergent to some $y \in X$. Since Γ is finer than Φ, y is a τ-cluster point of Φ. □

Similarly as for θ-regularity, there are numbers of simple examples of (τ,σ,ρ)-θ-regular spaces, including various modifications of regularity, compactness, local compactness, or paracompactness and we leave them to the reader. Note, for example, that a space $(\tau-\sigma)$ paracompact with respect to ρ is (τ,ρ,σ)-θ-regular.

Remark 2.4. One can easily check that (τ,σ,ρ)-θ-regularity is preserved by τ-closed subspaces if we consider the corresponding induced topologies on the subspace. On the other hand, as it is shown in [3], even F_σ-subspace of a compact (non-Hausdorff) space need not be countably θ-regular.

For a family $\{(X_\iota,\tau_\iota,\sigma_\iota,\rho_\iota) \mid \iota \in I\}$ denote by τ,σ,ρ the corresponding sum (product) topologies on $X = \sum_{\iota\in I} X_\iota$ $(X = \Pi_{\iota\in I} X_\iota)$. It is an easy exercise to prove that the topological sum X of $(\tau_\iota,\sigma_\iota,\rho_\iota)$-$\theta$-regular spaces X_ι, where $\iota \in I$, is (τ,σ,ρ)-θ-regular.

Theorem 2.5. *Let $X = \sum_{\iota\in I} X_\iota$ be the sum space for the family $\{(X_\iota,\tau_\iota,\sigma_\iota,\rho_\iota) \mid \iota \in I\}$ with the corresponding sum topologies τ,σ,ρ. Suppose that every X_ι is $(\tau_\iota,\sigma_\iota,\rho_\iota)$-$\theta$-regular. Then X is (τ,σ,ρ)-θ-regular.*

Theorem 2.6. *Let $X = \Pi_{\iota\in I} X_\iota$ be the product space for the family $\{(X_\iota,\tau_\iota,\sigma_\iota,\rho_\iota) \mid \iota \in I\}$ with the corresponding product topologies τ, σ, ρ. Suppose that every X_ι is $(\tau_\iota,\sigma_\iota,\rho_\iota)$-$\theta$-regular. Then X is (τ,σ,ρ)-θ-regular.*

Proof. Let Γ be an ultrafilter in X with (σ,ρ)-θ-limit $x = (x_\iota)_{\iota\in I} \in X$. Let $\pi_\iota : X \to X_\iota$ be the canonical projection. Then $\pi_\iota(\Gamma)$ is an ultrafilter on X_ι which $(\sigma_\iota,\rho_\iota)$-$\theta$-converges to x_ι. But X_ι is $(\tau_\iota,\sigma_\iota,\rho_\iota)$-$\theta$-regular. Hence, $\pi_\iota(\Gamma)$ τ_ι-converges to some $y_\iota \in X_\iota$, which implies that Γ τ-converges to $y = (y_\iota)_{\iota\in I}$. It follows that X is (τ,σ,ρ)-θ-regular. □

The productivity of θ-regularity proved in [4] by a different technique now follows as a corollary.

Definition 2.7. A bitopological space (X,τ,σ) is said to be *α-pairwise (countably) θ-regular* if X is (countably) (τ,τ,σ)-θ-regular and (countably) (σ,σ,τ)-θ-regular, *β-pairwise (countably) θ-regular* if X is (countably) (τ,σ,τ)-*θ-regular* and (countably) (σ,τ,σ)-θ-regular, *γ-pairwise (countably) θ-regular* if X is (countably) (τ,σ,σ)-*θ-regular* and (countably) (σ,τ,τ)-θ-regular and finally, *δ-pairwise (countably) θ-regular* if X is (countably) $(\tau\vee\sigma,\sigma,\tau\vee\sigma)$-$\theta$-regular and (countably) $(\tau\vee\sigma,\tau,\tau\vee\sigma)$-$\theta$-regular.

Remark 2.8. Using the characterization (i) in Theorem 2.1 and refining the open covers of the space several times, one can easily check that β- and γ-versions of pairwise θ-regularity are equivalent and imply the α-version, but not vice versa. Since every pairwise regular space obviously is α-pairwise θ-regular, the real line topologized by the intervals $(-\infty,p)$, $p \in \mathbb{R}$ for τ and (q,∞), $q \in \mathbb{R}$ for σ is a proper counterexample.

Remark 2.9. Observe that RR-pairwise paracompact and FHP-pairwise paracompact spaces are β-pairwise θ-regular and it can be easily seen that a β-pairwise θ-regular space has both topologies θ-regular.

However, for the following bitopological modifications of well-known Michael's theorem [5], only the β- and δ-versions of pairwise (countable) θ-regularity will be useful. In the proof of the next theorem, we slightly modify the technique used in [3].

Theorem 2.10. *Let σ_1, σ_2, σ_3, σ_4, be topologies on X. Let X be $(\sigma_1-\sigma_2)$ semiparacompact with respect to σ_3, $(\sigma_4-\sigma_3)$ semiparacompact with respect to σ_2 and countably $(\sigma_2,\sigma_4,\sigma_2)$-$\theta$-regular. Then X is $(\sigma_1-\sigma_2)$ paracompact with respect to σ_3.*

Proof. Let Ω be a σ_1-open cover of X. Since X is $(\sigma_1-\sigma_2)$ semiparacompact with respect to σ_3, it follows that Ω has a σ_2-open refinement, say $\Omega' = \bigcup_{i=1}^{\infty}\Omega_i$, where every Ω_i is a locally finite with respect to σ_3 family refining Ω.

Let $U_n = \bigcup\{U \mid U \in \Omega_i,\ i \le n\}$ for every $n \in \mathbb{N}$. The family $\{U_n\}_{n\in\mathbb{N}}$ is a countable σ_2-open increasing cover of X and since X is countably $(\sigma_2,\sigma_4,\sigma_2)$-$\theta$-regular, there exists a σ_4-open cover Φ of X whose σ_2-closures refine $\{U_n\}_{n\in\mathbb{N}}$. Because X is $(\sigma_4-\sigma_3)$

semiparacompact with respect to σ_2, Φ has a σ_3-open refinement, say $\Phi' = \bigcup_{i=1}^{\infty} \Phi_i$, consisting of families Φ_i which are locally finite with respect to σ_2. For every $n \in \mathbb{N}$, let

$$V_n = \bigcup \{B \mid B \in \Phi_i,\ \mathrm{cl}_{\sigma_2} B \subseteq U_j,\ i + j \le n\}. \tag{2.1}$$

The family $\{V_n\}_{n\in\mathbb{N}}$ is a σ_3-open increasing cover of X. Because the family $\bigcup_{i=1}^{n} \Phi_i$ is locally finite with respect to σ_2, we have $\mathrm{cl}_{\sigma_2} V_n \subseteq U_{n-1}$. Finally, for every $n \in \mathbb{N}$ and $U \in \Omega_n$, let

$$W_n(U) = U \setminus \mathrm{cl}_{\sigma_2} V_n. \tag{2.2}$$

It can be easily seen that the family $\Gamma = \{W_n(U) \mid n \in \mathbb{N},\ U \in \Omega_n\}$ is a σ_2-open cover of X which is a refinement of Ω locally finite with respect to σ_3. Indeed, for every $x \in X$ let $k \in \mathbb{N}$ be the least index such that $x \in U$ for some $U \in \Omega_k$. Since $\mathrm{cl}_{\sigma_2} V_k \subseteq U_{k-1}$, it follows that $x \in W_k(U)$. Hence Γ is a σ_2-open cover which, obviously, refines Ω. To see that Γ is locally finite with respect to σ_3, let $x \in X$ and let $m \in \mathbb{N}$ be any index such that $x \in V_m$. Because $\{V_n\}_{n\in\mathbb{N}}$ is an increasing family, we have $V_m \cap W_n(U) = \varnothing$ for every $n \ge m$, $U \in \Omega_n$.

But the family $\bigcup_{i=1}^{m} \Omega_i$ is locally finite with respect to σ_3. Let S be a σ_3-neighborhood of x, intersecting at most finitely many elements of $\bigcup_{i=1}^{m} \Omega_i$. Since for every $i = 1, 2, \dots, m$, $U \in \Omega_i$, we have $W_i(U) \subseteq U$, the set $S \cap V_m$ is a σ_3-neighborhood of x, meeting only finitely many sets of the cover Γ. Hence Γ is locally finite with respect to σ_3 and therefore X is $(\sigma_1 - \sigma_2)$ paracompact with respect to σ_3. □

In order to obtain a theorem for a bitopological space (X, τ_1, τ_2) from Theorem 2.10 it can be easily seen that there are only three meaningful possibilities for identifying the topologies σ_1, σ_2, σ_3, σ_4.

Case (i). $\tau_1 = \sigma_1 = \sigma_4$ and $\tau_2 = \sigma_2 = \sigma_3$.

Corollary 2.11. *Let X be countably (τ_2, τ_1, τ_2)-θ-regular and $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_2. Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_2.*

Corollary 2.12. *Let X be a bitopological space. Then X is FHP-pairwise paracompact if and only if X is β-pairwise countably θ-regular and FHP-pairwise semiparacompact.*

Proof. It is sufficient to use the previous corollary twice. □

Note that Raghavan and Reilly stated [7, Theorem 3.9] from which it would follow that a pairwise regular δ-pairwise semiparacompact space is δ-pairwise paracompact. Unfortunately, (iv)⟹(i) in the proof of this theorem is not correct. The authors used [1, Theorem 1.5, page 162] in the proof. However, the assumptions of the theorem are not completely satisfied. They tried to expand a locally finite cover $\mathcal{V}$ to the open one using a closed cover such that every its element meets only finitely many members of $\mathcal{V}$. However, in general the used closed cover is not locally finite or at least closure preserving. That is not sufficient for the expansion, as the following example shows.

Example 2.13. Let $C = \mathbb{N} \times \langle -1, 1)$, $B = \mathbb{N} \times (0, 1)$, and $A = \mathbb{N} \times \langle -1, 0\rangle$. We consider the Euclidean topology on C induced from the real plane and let $X = C \cup \{y \mid$

$\mathcal{y}$ is a nonconvergent ultra-closed filter in C, $B \in \mathcal{y}\}$. Let $S(U) = U \cup \{\mathcal{y} \mid \mathcal{y} \in X \setminus C$, $U \in \mathcal{y}\}$ for any $U \subseteq C$ open in C. Of course, X is a subspace of the Wallman compactification ωC and the sets $S(U)$ constitute a topology base for X. Since C is normal, ωC is Hausdorff and hence X is a $T_{3.5}$ space. Denote $A_n = \{n\} \times \langle -1, 0\rangle$. The family $\Omega = \{S(B), A_1, A_2, A_3, \dots\}$ is a locally finite cover of X, which has no open locally finite extension.

Indeed, suppose that there are some open U_n with $A_n \subseteq U_n$ for $n \in \mathbb{N}$. Then every U_n must meet $B_n = \{n\} \times (0,1)$. Choose $x_n \in U_n \cap B_n$ for each $n \in \mathbb{N}$. Let $F_n = \{x_n, x_{n+1}, \dots\}$. Since the sequence $x_1, x_2, \dots$ has no cluster point in C, the collection $\Phi = \{F_n \mid n = 1, 2, \dots\}$ is a closed filter base in C with no cluster point in C. It follows that there is a non-convergent ultra-closed filter, say $\mathcal{y} \in \omega C$, finer than Φ. But $F_1 \subseteq B$ and since $F_1 \in \Phi \subseteq \mathcal{y}$, $B \in \mathcal{y}$. Hence $\mathcal{y} \in X$. Let W be any open neighborhood of $\mathcal{y}$ in X. There is some V open in C with $\mathcal{y} \in S(V) \subseteq W$. Then $V \in \mathcal{y}$ and hence $V \cap F_n \neq \emptyset$ for every $n \in \mathbb{N}$. Thus for any fixed $m \in \mathbb{N}$ there exists $n \geq m$ such that $x_n \in V \subseteq S(V) \subseteq W$ and therefore W intersects infinitely many elements of $\{U_n \mid n = 1, 2, \dots\}$. Hence Ω cannot be expanded to an open locally finite cover.

On the other hand, the previous example does not refute Raghavan-Reilly's theorem, which still remains open as a question. With a different modification of the concept of pairwise regularity the theorem is correct.

Corollary 2.14. *Let X be δ-pairwise countably θ-regular. Then X is δ-pairwise paracompact if and only if X is δ-pairwise semiparacompact.*

Proof. Since X is countably $(\tau_1 \vee \tau_2, \tau_1, \tau_1 \vee \tau_2)$-$\theta$-regular and $(\tau_1 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$, it follows that X is $(\tau_1 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$ by Corollary 2.11. But X is also countably $(\tau_1 \vee \tau_2, \tau_2, \tau_1 \vee \tau_2)$-$\theta$-regular and $(\tau_2 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$ which implies, also by Corollary 2.11, that X is $(\tau_2 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$. Hence X is δ-pairwise paracompact in topologies τ_1, τ_2. □

Remark 2.15. Note that the space X constructed in Example 2.13 is $T_{3.5}$ but not normal—the sets $A, X \setminus C$ are closed, pairwise disjoint but they have no disjoint neighborhoods.

Case (ii). $\tau_1 = \sigma_1 = \sigma_2$ and $\tau_2 = \sigma_3 = \sigma_4$.

Corollary 2.16. *Let X be a bitopological space. Then X is RR-pairwise paracompact if and only if X is β-pairwise countably θ-regular and RR-pairwise semiparacompact.*

Case (iii). $\tau_1 = \sigma_1 = \sigma_3$ and $\tau_2 = \sigma_2 = \sigma_4$.

Corollary 2.17. *Let τ_1, τ_2 be countably θ-regular topologies of X. Suppose that X is $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_1, and $(\tau_2 - \tau_1)$ semiparacompact with respect to τ_2. Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_1 and $(\tau_2 - \tau_1)$ paracompact with respect to τ_2.*

Finally, remark that modifying properly the concept of $\sum$-space for bitopological spaces, combining Theorem 2.6 and the corollaries of Theorem 2.10 similar results as in [4] (see [6, Nagami's theorem]) for the countable product of paracompact $\sum$-spaces without necessity of Hausdorff-type separation are also possible.

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