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ALMOST AUTOMORPHIC SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS IN BANACH SPACES

GASTON MANDATA N'GUEREKATA

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ABSTRACT. We discuss the conditions under which bounded solutions of the evolution equation $x'(t) = Ax(t) + f(t)$ in a Banach space are almost automorphic whenever $f(t)$ is almost automorphic and A generates a C_0-group of strongly continuous operators. We also give a result for asymptotically almost automorphic solutions for the more general case of $x'(t) = Ax(t) + f(t,x(t))$.

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1. Introduction. Let A generate a C_0-group of strongly continuous operators $T(t)$, $t \in \mathbb{R}$ on a Banach space X. Let $f \in L^\infty(\mathbb{R};X)$. A basic unsolved problem is: what is the structure of bounded (on $\mathbb{R}$) mild solutions of $x'(t) = Ax(t) + f(t)$? Classically results go back to Ordinary Differential Equations (when dimension of X is finite), and one sought solutions $x(t)$ such that $x(t) - y(t) \to 0$ as $t \to \infty$, when either $y(t)$ is a constant or a periodic function of time. In the evolution context of $x' = Ax + f$, much has been written on asymptotically constant or periodic solutions. Several authors extended these ideas to almost periodic solutions (when f is almost periodic). Our main result (Theorem 1.6) is inspired by the interesting work of Goldstein [3]. We are actually concerned with the more general case of almost automorphic, and when bounded solutions are almost automorphic. We also give a new result (Theorem 1.7) concerning mild solutions of the equation $x'(t) = Ax(t) + f(t,x(t))$ which approach almost automorphic functions at infinity under specific conditions on the function $f(t,x)$. See also [6] for another comparable situation.

Let X be a Banach space equipped with the topology norm and $\mathbb{R} = (-\infty,\infty)$ the set of real numbers. Let us first recall some definitions.

DEFINITION 1.1 (Bochner). A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if and only if, from any sequence of real numbers $(s'_n)_{n=1}^\infty$, we can subtract a subsequence $(s_n)_{n=1}^\infty$ such that: $\lim_{n\to\infty} f(t+s_n) = g(t)$ exists for each real number t, and $\lim_{n\to\infty} g(t-s_n) = f(t)$ for each t.

DEFINITION 1.2 [4]. A continuous function $f : \mathbb{R}^+ \to X$ is said to be asymptotically almost automorphic if and only if there exists an almost automorphic function $g : \mathbb{R} \to X$ and a continuous function $h : \mathbb{R}^+ \to X$ with $\lim_{t\to\infty} \|h(t)\| = 0$ and such that $f(t) = g(t) + h(t)$ for each $t \in \mathbb{R}^+$.

Definition 1.3. A Banach space X is said to be perfect if and only if every bounded function $u:\mathbb{R}\to X$ with an almost automorphic derivative $u'(t)$ is necessarily almost automorphic.

Remark 1.4. Uniformly convex Banach spaces are nice examples of perfect Banach spaces (see [10, Theorem 1.4]).

We consider the evolution equation

$$x'(t)=Ax(t)+f(t),\quad t\in\mathbb{R}. \tag{1.1}$$

Theorem 1.5. *Let X be a perfect Banach space. Let A be a bounded linear operator $X\to X$ and $f:\mathbb{R}\to X$ an almost automorphic function. Then any bounded strong solution of (1.1) is almost automorphic if we assume that there exists a finite-dimensional subspace X_1 of X such that*

(α) $Ax(0)\in X_1$,

(β) $(e^{tA}-I)f(s)\in X_1$ *for any* $s,t\in\mathbb{R}$,

(γ) $e^{tA}u\in X_1$ *for any* $t\in\mathbb{R}$ *and for any* $u\in X_1$.

Proof. Let P be the projection of X onto X_1; such P always exists (cf. [7]) and possesses the following properties:

(1) $X=X_1\oplus\ker(P)$, where $\ker(P)$ is the kernel of the operator P,

(2) P is bounded on X.

If we put $Q=I-P$, then it is easy to verify that $Q^2=Q$ on X and $Qu=0$ for any $u\in X_1$. Now if $x(t)$ is a bounded solution of (1.1), then we can write it as

$$x(t)=x_1(t)+x_2(t) \tag{1.2}$$

with $x_1(t)=Px(t)\in X_1$ and $x_2(t)=Qx(t)\in\ker(P)$.

Since $x(t)$ is bounded on $\mathbb{R}$, it is clear that both $x_1(t)$ and $x_2(t)$ are also bounded on $\mathbb{R}$. On the other hand, we have

$$x'(t)=x_1'(t)+x_2'(t)=Ax_1(t)+Ax_2(t)+Pf(t)+Qf(t),\quad t\in\mathbb{R}. \tag{1.3}$$

But $x(t)$ has the well-known Lagrange representation:

$$\begin{aligned} x(t)&=e^{tA}x(0)+\int_0^t e^{(t-s)A}f(s)\,ds\\ &=e^{tA}x(0)+\int_0^t f(s)\,ds+\int_0^t\left(e^{(t-s)A}-I\right)f(s)\,ds. \end{aligned} \tag{1.4}$$

By assumption (β), we deduce that $\int_0^t(e^{(t-s)A}-I)f(s)ds$ is in X_1, so that if we apply Q to both sides of (1.4), we get

$$x_2(t)=Qe^{tA}x(0)+Q\int_0^t f(s)\,ds=Qe^{tA}x(0)+\int_0^t Qf(s)\,ds, \tag{1.5}$$

consequently

$$x_2'(t)=Qe^{tA}Ax(0)+Qf(t)=Qf(t) \tag{1.6}$$

using conditions (α) and (γ).

It is clear that $Qf(t)$ and thus $x_2'(t)$ is almost automorphic (see [9, page 586]). Since $x_2(t)$ is bounded, then it is almost automorphic for we are in a perfect Banach space.

Now if we apply P to both sides of (1.3), we get in the finite-dimensional space X_1 the differential equation

$$x_1'(t) = PAx_1(t) + PAx_2(t) + P^2 f(t) + PQf(t), \quad t \in \mathbb{R}. \tag{1.7}$$

Since the function $g(t) \equiv P^2 f(t) + PQf(t)$ is almost automorphic and PA is a bounded linear operator, we deduce that $x_1(t)$ is almost automorphic [9, Theorem 3]. Finally, $x(t)$ is almost automorphic as the sum of two almost automorphic functions. □

Theorem 1.5 can be generalized to the case of unbounded operator A as follows.

Theorem 1.6. *In a perfect Banach space X, let A generate a C_0-group of strongly continuous linear operators $T(t)$, $t \in \mathbb{R}$. Assume that there exists a finite-dimensional subspace X_1 of X such that:*

(α) $Ax(0) \in X_1$,

(β') $(T(t) - I)f(s) \in X_1$ *for any* $s, t \in \mathbb{R}$,

(γ) $T(t)u \in X_1$ *for any* $t \in \mathbb{R}$ *and any* $u \in X_1$.

Then every bounded solution of (1.1) is almost automorphic.

Proof. We just follow the proof of Theorem 1.5 with the appropriate modifications. Here solutions are written as $x(t) = T(t)x(0) + \int_0^t T(t-s)f(s)ds$. □

We return now to a general (not necessarily perfect) Banach space X. We state and prove the following theorem.

Theorem 1.7. *Let A be a (possibly unbounded) linear operator which is the generator of a C_0-group of strongly continuous linear operators $T(t)$, $t \in \mathbb{R}$ such that $T(t)x : \mathbb{R} \to X$ is almost automorphic for each $x \in X$. Consider the differential equation*

$$x'(t) = Ax(t) + f(t, x(t)), \tag{1.8}$$

where $f(t,x) : \mathbb{R} \times X \to X$ is strongly continuous with respect to jointly t and x and such that $\|f(t,x) - f(t,y)\| \le L\|x - y\|$ for any $t \in \mathbb{R}$, $x, y \in X$, and $\int_0^\infty \|f(t,0)\|dt < \infty$.

Then every mild solution $x(t)$ of (1.8) with $\int_0^\infty \|x(t)\|dt < \infty$ is asymptotically almost automorphic.

Proof. Let $x : \mathbb{R}^+ \to X$ be a mild solution of (1.8). Then we have

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(s, x(s))\, ds. \tag{1.9}$$

We claim that $\int_0^\infty T(-s)f(s,x(s))ds$ exists in X (in Bochner's sense). Indeed, since $T(t)$ is almost automorphic for each $x \in X$, then

$$\sup_{t \in \mathbb{R}} \|T(t)x\| < \infty \quad \text{for each } x \in X. \tag{1.10}$$

Consequently

$$\sup_{t \in \mathbb{R}} \|T(t)\| = M < \infty, \tag{1.11}$$

by the uniform boundedness principle. Let us write

$$\int_0^\infty T(-s)f(s,x(s))\,ds = \int_0^\infty T(-s)\big(f(s,x(s)) - f(s,0)\big)\,ds + \int_0^\infty T(-s)f(s,0)\,ds, \tag{1.12}$$

then we get the inequality

$$\left\|\int_0^\infty T(-s)f(s,x(s))\,ds\right\| \le M\left(L\int_0^\infty \|x(s)\|\,ds + \int_0^\infty \|f(s,0)\|\,ds\right) < \infty. \tag{1.13}$$

Now the continuous function $F : \mathbb{R} \to X$ defined by

$$F(t) = \int_0^\infty T(t-s)f(s,x(s))\,ds = T(t)\int_0^\infty T(-s)f(s,x(s))\,ds \tag{1.14}$$

is almost automorphic; therefore $V(t) = T(t)x(0) + F(t)$ is also almost automorphic. Let us consider the continuous function $W : \mathbb{R}^+ \to X$

$$W(t) = -\int_t^\infty T(t-s)f(s,x(s))\,ds. \tag{1.15}$$

If we use the same computation as for $F(t)$ in (1.14), we get

$$\|W(t)\| \le M\left(L\int_t^\infty \|x(s)\,ds\| + \int_t^\infty \|f(s,0)\,ds\|\right) \tag{1.16}$$

which shows that $\lim_{t\to\infty}\|W(t)\| = 0$.

Finally $x(t) = V(t) + W(t)$, $t \in \mathbb{R}^+$ is asymptotically almost automorphic. □

Remark 1.8. (1) An example of Theorem 1.5 (occurring in Sturm-Liouville theory, for instance) is when X is a Hilbert space and $A\varphi_n = \lambda_n\varphi_n$ for $\{\varphi_n : n = 1,2,\ldots\}$ an orthonormal basis and $|\mathrm{Re}(\lambda_n)| \le M$ for all n. For X_1, one may take $X_1 = \mathrm{span}\{\varphi_1,\ldots,\varphi_N\}$ (for any N) and assume $f \in L^\infty(\mathbb{R}, X_1)$.

(2) An example of operator A satisfying the hypothesis of Theorem 1.7 is the above example with $A^* = -A$, i.e., $M = 0$.

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N'Guerekata: Department of Mathematics, Morgan State University, Cold Spring Lane and Hillen Road, Baltimore, MD 21251, USA