A NOTE ON \( M \)-IDEALS IN CERTAIN ALGEBRAS OF OPERATORS

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Abstract. Let \( X = (\sum_{n=1}^{\infty} \ell_{1}^{p})_p, \; p > 1 \). In this paper, we investigate \( M \) ideals which are also ideals in \( L(X) \), the algebra of all bounded linear operators on \( X \). We show that \( K(X) \), the ideal of compact operators on \( X \) is the only proper closed ideal in \( L(X) \) which is both an ideal and an \( M \)-ideal in \( L(X) \).

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1. Introduction. Since Alfsen and Effros [1, 2] introduced the notion of an \( M \)-ideal in a Banach space, many authors have studied \( M \)-ideals in algebras of operators. An interesting problem has been characterizing and finding those Banach spaces \( X \) for which \( K(X) \), the space of all compact linear operators on \( X \), is an \( M \)-ideal in \( L(X) \), the space of all continuous linear operators on \( X \) [4, 8, 9, 11, 12].

It is known that if \( X \) is a Hilbert space, \( \ell_{p} (1 < p < \infty) \) or \( c_{0} \), then \( K(X) \) is an \( M \)-ideal in \( L(X) \) [6, 8, 12] while \( K(\ell_{1}) \) and \( K(\ell_{\infty}) \) are not \( M \)-ideals in the corresponding spaces of operators [12]. Smith and Ward [12] proved that \( M \)-ideals in a complex Banach algebra with identity are subalgebras and that they are two-sided algebraic ideals if the algebra is commutative. They also proved that \( M \)-ideals in a C*-algebra are exactly the two-sided ideals [12]. Later, Cho and Johnson [5] proved that if \( X \) is a uniformly convex Banach space, then every \( M \)-ideal in \( L(X) \) is a left ideal, and if \( X^* \) is also uniformly convex, then every \( M \)-ideal in \( L(X) \) is a two-sided ideal in \( L(X) \).

Flinn [7], and Smith and Ward [13] proved that \( K(\ell_{p}) \) is the only nontrivial \( M \)-ideal in \( L(\ell_{p}) \) for \( 1 < p < \infty \). Kalton and Werner [10] proved that if \( 1 < p, q < \infty, X = (\sum_{n=1}^{\infty} \ell_{q}^{n})_p \) with complex scalars, then \( K(X) \) is the only nontrivial \( M \)-ideal in \( L(X) \). In their proof of this fact, Kalton and Werner [13] used the uniform convexity of \( X \) and \( X^* \). In this case, \( M \)-ideals in \( L(X) \) are two-sided closed ideals in \( L(X) \) [5].

In this paper, we investigate \( M \)-ideals which are also ideals in \( L(X) \) for \( X = (\sum_{n=1}^{\infty} \ell_{1}^{n})_p, \; 1 < p < \infty \). In our case, neither \( X \) nor \( X^* \) is uniformly convex. Therefore, no relationships between \( M \)-ideals and algebraic ideals in \( L(X) \) seem to be known. But still we can use Kalton and Werner’s proof in [10] without using uniform convexity of \( X \) and \( X^* \) to prove that \( K(X) \) is the only nontrivial \( M \)-ideal in \( L(X) \) which is also a closed ideal in \( L(X) \) (Theorem 3.3). By duality we have the same conclusion for the space \( (\sum_{n=1}^{\infty} \ell_{\infty}^{n})_p, \; 1 < p < \infty \).

2. Preliminaries. A closed subspace \( J \) of a Banach space \( X \) is said to be an \( L \)-summand (respectively, \( M \)-summand) if there exists a closed subspace \( J' \) of \( X \)
such that $X$ is an algebraic direct sum $X = J \oplus J'$ and satisfies a norm condition
\[ \|j + j'\| = \|j\| + \|j'\| \] (respectively, $\|j + j'\| = \max\{\|j\|, \|j'\|\}$) for all $j \in J$ and $j' \in J'$.
In this case, we write $X = J \oplus_1 J'$ (respectively, $X = J \oplus_\infty J'$) and the projection $P$ on $X$
with rang $J$ is called an $L$-projection (respectively, an $M$-projection). A closed subspace $J$
of a Banach space $X$ is an $M$-ideal in $X$ if the annihilator $J^\perp$ of $J$ is an $L$-summand in $X^*$.

Let $A$ be a complex Banach algebra with identity $e$. The state space $S$ of $A$ is defined
to be $\{\phi \in A^* : \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is said to be Hermitian if $\|e^{ij}\| = 1$
for all real number $\lambda$. Equivalently, $h$ is Hermitian if and only if $\phi(h)$ is real for every
$\phi \in S$ \cite[page 46]{1}.

In what follows, $Z$ always denote a complex Banach space $(\sum_{n=1}^\infty \ell_1^n)_p$, the $\ell_p$-sum
of $\ell_1^n$s for $1 < p < \infty$. For each $n$, let $\{e_{nl}\}_{l=1}^n$ be the standard basis of $\ell_1^n$. Then these
bases string together to form the standard basis $\{e_n\}_{n=1}^\infty$ of $Z$ and each $T \in L(Z)$ has
a matrix representation with respect to $\{e_n\}_{n=1}^\infty$. If $T \in L(Z)$ has the matrix whose
$(i, j)$-entry is $t_{ij}$, then we can write $T = \sum_{i,j=1}^n t_{ij} e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank 1 map
sending $e_j$ to $e_i$. Observe that $T(e_j)$ forms the $j$th column vector of the matrix of $T$
and $\|Te_j\| \leq \|T\|$ for all $j = 1, 2, \ldots$. If the matrix of $T$ has at most one nonzero entry
in each row and column, then $\|T\|$ is the $l_\infty$-norm of the sequence of nonzero entries.

A number of facts which hold in $L(\ell_p), 1 < p < \infty$, still hold in $L(Z)$. If the matrix of $T \in L(Z)$ is a diagonal matrix $(t_{ij})$ with real diagonal entries, then for each real $\lambda$
the matrix of $e^{i\lambda T}$ is also a diagonal matrix with diagonal matrix entries $e^{i\lambda t_{ii}}$. Thus
$T \in L(Z)$ is Hermitian if the matrix $T$ is a diagonal matrix with real entries.

Flinn \cite{7} proved that if $M$ is an $M$-ideal in $L(\ell_p), 1 < p < \infty$ and $h$ is a Hermitian element in $L(\ell_p)$ with $h^2 = I$, then $hM \subseteq M$ and $Mh \subseteq M$. From this he proved that if $h$
is an $M$-ideal matrix with real entries, then $hM \subseteq M$ and $Mh \subseteq M$. His proof is
valid for $Z$ in place of $\ell_p$. Thus we have the following.

**Lemma 2.1.** If $M$ is an $M$-ideal in $L(Z)$ and $h \in L(Z)$ is a diagonal matrix with real
entries, then $hM \subseteq M$ and $Mh \subseteq M$.

The $M$-ideal structure of $L(X)$ for $X = (\sum_{n=1}^\infty \ell_1^n)_p, 1 < p, q < \infty$ was studied by
Kalton and Werner \cite{10}. Some of their proofs for $X$ are still good for $Z$. One of them
is the following.

**Lemma 2.2.** There is a constant $C$ such that, whenever $(k_n)$ is a sequence of positive
integers with $\limsup k_n = \infty$, then $(\sum_{n=1}^\infty \ell_1^{k_n})_p$ is $C$-isomorphic to $(\sum_{n=1}^\infty \ell_1^n)_p$.

**Proof.** See proof of Lemma 3.1 of \cite{10}.

We recall that a Banach space $X$ is $C$-isomorphic to a Banach space $Y$ if there exists
an isomorphism $T$ form $X$ onto $Y$ such that
\[
\frac{1}{C} \|x\| \leq \|Tx\| \leq C \|x\| \tag{2.1}
\]
for every $x \in X$. We use the following lemma which is due to Kalton and Werner \cite{10}.

**Lemma 2.3** \cite{10}. Let $X$ be a Banach space, $\mathcal{F} \subset L(X)$ be a two-sided ideal, and $P$
a projection onto a complemented subspace $E$ of $X$ which is isomorphic to $X$.

(a) If $P \in \mathcal{F}$, then $\mathcal{F} = L(X)$. 

If $E$ is $C$-isomorphic with $X$ and $\mathcal{T}$ contains an operator $T$ with $\|T - P\| < (C\|P\|^{-1})$, then $\mathcal{T} = L(X)$.

3. $M$-ideals in $L((\sum_{n=1}^{\infty} \ell_1^n)_p)$. A matrix carpentry used by Flinn [7] to characterize the $M$-ideal structure in $L(\ell_p)$ can be used to some extent in our case $Z = (\sum_{n=1}^{\infty} \ell_1^n)_p$.

The proof of the following lemma is really a minor modification of Flinn’s proof in [7].

**Lemma 3.1.** If $M$ is a nontrivial $M$-ideal in $L(Z)$, then $K(Z) \subseteq M$.

**Sketch of the proof.** Let us call two positive integers $i$ and $j$ are in the same block if $n(n + 1)/2 < i, j \leq (n + 1)(n + 2)/2$ for some $n$. Using Lemma 2.1, we can follow Flinn’s proof of the second corollary to Lemma 1 in [7]. The only modification is the following: to prove $2^{1/q} < |t_{pl} + t_{kl}| \leq 2^{1/q}$, we consider two cases. If $p$ and $k$ are in a different block, Flinn’s proof just run through. If $p$ and $k$ are in the same block, then $2^{1/q} < |t_{pl} + t_{kl}| \leq \|T(e_l)\| \leq 2^{1/q}$.

The proof of the following lemma is contained in the proof of Theorem 3.3 in [10].

**Lemma 3.2.** If $\mathcal{T}$ is a closed ideal strictly containing $K(Z)$ then $\mathcal{T}$ contains all the operators which factor through $\ell_p$.

The proof of the following theorem is a modification of that of Kalton and Werner [10]. Here we can go around the use of uniform convexity.

**Theorem 3.3.** If $\mathcal{T}$ is a closed ideal and also an $M$-ideal in $L(Z)$ strictly containing $K(Z)$, then $\mathcal{T} = L(Z)$.

**Proof.** We recall that the standard basis $\{e_{nl}\}_{n=1}^{\infty}$ of $\ell_1^n$ string together to form the standard basis $\{e_n\}_{n=1}^{\infty}$ of $Z$. If $\{e_n^*\}_{n=1}^{\infty}$ is the standard basis of $\ell_p$, then the map $e_n - e_n^*$ gives a contraction from $Z$ to $\ell_p$. Since $E = \text{span}\{e_{nl}\}_{n=1}^{\infty}$ is isometric to $\ell_p$, there exists a norm one operator $A$ from $Z$ to $E$ carrying $e_n$ to $e_{nl}$ via $e_n^*$. Thus $A$ factors through $\ell_p$. By Lemma 3.2, $A \in \mathcal{T}$.

Since $\mathcal{T}$ is also an $M$-ideal, by Proposition 2.3 in [14], there exists a net $(H_\alpha) \subseteq \mathcal{T}$ such that

$$\limsup \|\pm A + (\text{Id} - H_\alpha)\| = 1. \quad (3.1)$$

To simplify subsequent calculations, let us write the standard basis of $Z$ as $\{e_{nl} : n \in \mathbb{N}, 1 \leq l \leq n\}$ and let $\{e_{nl}^* : n \in \mathbb{N}, 1 \leq l \leq n\}$ be the corresponding biorthogonal functionals. Then $Ae_{nl} = e_{ml}$, where $m = (n - 1)n/2 + l$.

Given $0 < \varepsilon < 1$,

$$\max_\pm \|\pm A + (\text{Id} - H_\alpha)\| < 1 + \varepsilon \quad (3.2)$$

for infinitely many $\alpha$‘s. For such an $\alpha$ and every $e_{nl}$,

$$\max_\pm \|\pm Ae_{nl} - (\text{Id} - H_\alpha)e_{nl}\| < 1 + \varepsilon. \quad (3.3)$$
Put $\alpha_{kj} = e^*_k (\Id - H_\alpha)e_{nl}$. Then,
\[
\max \| \pm Ae_{nl} + (\Id - H_\alpha)e_{nl} \|_p = \max \| \pm e_{m1} - (\Id - H_\alpha)e_{nl} \|_p \\
= \left( \max \| \alpha_{m1} \pm 1 + |\alpha_{m2}| + \cdots + |\alpha_{mm}| \right)^p + \sum_{k \neq m} \left( \sum_{j=1}^k |\alpha_{kj}| \right)^p < (1 + \epsilon)^p.
\] (3.4)

Since $\max \| \alpha_{m1} \pm 1 \|_p \geq 1$, it follows that $\sum_{k \neq m} \left( \sum_{j=1}^k |\alpha_{kj}| \right)^p < (1 + \epsilon)^p - 1$ and $|\alpha_{m2}| + \cdots + |\alpha_{mm}| < \epsilon$. Since $\sqrt{1 + |\alpha_{m1}|^2} \leq \max \| \alpha_{m1} \pm 1 \|_p < 1 + \epsilon$, $|\alpha_{m1}| < \sqrt{2\epsilon + \epsilon^2} < 2\sqrt{\epsilon}$. Thus, $\|(\Id - H_\alpha)e_{nl}\| \leq \left( (3\sqrt{\epsilon})^p + (1 + \epsilon)^p - 1 \right)^{1/p} - 0$ as $\epsilon \to 0$ uniformly in $n$ and $l$. It follows that, for any $n$,
\[
\|P_n (\Id - H_\alpha)P_n\| \leq \|P_n (\Id - H_\alpha)j_n\| \leq \left( (3\sqrt{\epsilon})^p + (1 + \epsilon)^p - 1 \right)^{1/p},
\] (3.5)

where $P_n$ is the projection on $Z$ with range $\ell^n_1 \subseteq Z$ and $j_n$ is the canonical injection of $\ell^n_1$ into $Z$.

By Lemma 3.2 in [10], there exists a sequence $(k_n)$ such that, for the canonical projection $P$ from $Z$ onto $(\sum_{n=1}^\infty \ell^{kn}_1)_p$,
\[
\|P - PH_\alpha P\| = \|P (\Id - H_\alpha)P\| < 3 \left( (3\sqrt{\epsilon})^p + (1 + \epsilon)^p - 1 \right)^{1/p}.
\] (3.6)

Since $PH_\alpha P \in \mathcal{T}$ and $\epsilon > 0$ is arbitrary small, by Lemmas 2.2 and 2.3, $\mathcal{T} = L(Z)$.

From Lemma 3.1 and Theorem 3.3, we have the following.

**COROLLARY 3.4.** If $\mathcal{T}$ is a proper ideal and also an $M$-ideal in $L(Z)$, then $\mathcal{T} = K(Z)$.

**REMARK.** By duality, all the lemmas, Theorem 3.3 and Corollary 3.4 hold with $Z^* = (\sum_{n=1}^\infty \ell^n_1)_p$, $1 < p < \infty$, in place of $Z$.

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**REFERENCES**


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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