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# MARCINKIEWICZ-TYPE STRONG LAW OF LARGE NUMBERS FOR DOUBLE ARRAYS OF PAIRWISE INDEPENDENT RANDOM VARIABLES

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ABSTRACT. Let $\{X_{ij}\}$ be a double sequence of pairwise independent random variables. If $P\{|X_{mn}| \geq t\} \leq P\{|X| \geq t\}$ for all nonnegative real numbers $t$ and $E|X|^p(\log^+|X|)^3 < \infty$, for $1 < p < 2$, then we prove that

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})}{(mn)^{1/p}} \longrightarrow 0 \quad \text{a.s.} \quad \text{as } m \vee n \longrightarrow \infty. \tag{0.1}$$

Under the weak condition of $E|X|^p \log^+|X| < \infty$, it converges to 0 in $L^1$. And the results can be generalized to an $r$-dimensional array of random variables under the conditions $E|X|^p(\log^+|X|)^{r+1} < \infty$, $E|X|^p(\log^+|X|)^{r-1} < \infty$, respectively, thus, extending Choi and Sung's result [1] of the one-dimensional case.

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**1. Introduction.** Etemadi [3] extended the classical law of large numbers for i.i.d. random variables to the case where the random variables are pairwise i.i.d., i.e., if $\{X_n\}$ is a sequence of pairwise i.i.d. random variables with $E|X_1| < \infty$, then

$$\frac{\sum_{i=1}^{n}(X_i-EX_i)}{n} \longrightarrow 0 \quad \text{a.s.} \tag{1.1}$$

In 1985, Choi and Sung [1] have shown that if $\{X_n\}$ are pairwise independent and are dominated in distribution by a random variable $X$ with $E|X|^p(\log^+|X|)^2 < \infty$, $1 < p < 2$, then $\frac{\sum_{i=1}^{n}(X_i-EX_i)}{n^{1/p}} \longrightarrow 0$ a.s. In addition, if $E|X|^p < \infty$, then it converges to 0 in $L^1$.

For a double sequence $\{X_{ij}\}$ of pairwise i.i.d. random variables, also Etemadi [3] proved that if $E|X_{11}|\log^+|X_{11}| < \infty$, then

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-EX_{ij})}{mn} \longrightarrow 0 \quad \text{a.s.} \quad \text{as } m \vee n \longrightarrow \infty. \tag{1.2}$$

Now, we are interested in the extension of Choi and Sung's result of the one-dimensional case to a multi-dimensional array of pairwise independent random variables, which is established in the next section.

**2. Main results.** Let $\{X_{ij}\}$ be a double sequence of random variables and let $X'_{ij} = X_{ij}I\{|X_{ij}| \leq (ij)^{1/p}\}$, $X''_{ij} = X_{ij}I\{|X_{ij}| > (ij)^{1/p}\}$ for $1 < p < 2$. Throughout this paper,

$c$ denotes an unimportant positive constant which is allowed to change and $d_k$ the number of all divisors of integer $k$.

To prove the main theorem, we need the following lemmas.

**Lemma 2.1.** *Let $\{X_{ij}\}$ be a double sequence of pairwise independent random variables. If $P\{|X_{mn}| \ge t\} \le P\{|X| \ge t\}$ for all nonnegative real numbers $t$, then*

$$\text{(a)} \qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E\left|X'_{ij}\right|^2}{(ij)^{2/p}} \le cE|X|^p\log^+|X|,$$
$$\text{(b)} \qquad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E\left|X''_{ij}\right|}{(ij)^{1/p}} \le cE|X|^p\log^+|X| \quad \text{for } 1<p<2. \tag{2.1}$$

**Proof.** The estimation of $E|X'_{ij}|^2$ is given by

$$\begin{aligned} E\left|X'_{ij}\right|^2 &\le \int_0^{(ij)^{2/p}} P\left(\left|X_{ij}\right|^2 \ge t\right)dt \\ &\le \int_0^{(ij)^{2/p}} P\left(|X|^2 \ge t\right)dt \\ &= \int_0^{(ij)^{2/p}} \left\{P\left(t \le |X|^2 < (ij)^{2/p}\right) + P\left((ij)^{2/p} \le |X|^2\right)\right\}dt \\ &= \int_0^{(ij)^{1/p}} x^2\,dF(x) + (ij)^{2/p}P\left((ij)^{2/p} \le |X|^2\right), \end{aligned} \tag{2.2}$$

where $F(x)$ is the distribution of $X$. If we use the fact that $\sum_{k=i+1}^{\infty} d_k/k^{2/p} = O(\log i/(i+1)^{2/p-1})$, we obtain

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{(ij)^{2/p}}\int_0^{(ij)^{1/p}} x^2\,dF(x) &\le c\sum_{k=1}^{\infty}\frac{d_k}{k^{2/p}}\int_0^{k^{1/p}} x^2\,dF(x) \\ &\le c\sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty}\frac{d_k}{k^{2/p}}\right)\int_{i^{1/p}}^{(i+1)^{1/p}} x^2\,dF(x) \\ &\le c\sum_{i=0}^{\infty}\frac{\log i}{(i+1)^{2/p-1}}\int_{i^{1/p}}^{(i+1)^{1/p}} x^2\,dF(x) \\ &\le cE|X|^p\log^+|X| < \infty. \end{aligned} \tag{2.3}$$

And

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P\left((ij)^{2/p} \le |X|^2\right) &= \sum_{k=1}^{\infty} d_k P\left(k \le |X|^p\right) \\ &= \sum_{k=1}^{\infty}\left(\sum_{j=1}^{k} d_j\right)P\left(k \le |X|^p < k+1\right) \\ &= c\sum_{k=1}^{\infty} k\log k P\left(k \le |X|^p < k+1\right) \\ &\le cE|X|^p\log^+|X| < \infty, \end{aligned} \tag{2.4}$$

where we use the fact that $\sum_{k=1}^{n} d_k = O(n\log n)$. It follows that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{E|X'_{ij}|^2}{(ij)^{2/p}} < \infty, \quad \text{which proves (a).} \tag{2.5}$$

By the fact that $\sum_{k=1}^{n} d_k/k^{1/p} = O(n^{1-(1/p)}\log n)$, we can obtain (b) by the same method. □

The following lemma is a two parameter analog of [5, Lem. 3.6.1a].

**Lemma 2.2.** *Let $\{X_{ij}\}$ be a double sequence of pairwise independent random variables with $EX_{ij} = 0$, and let $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$. Then*

$$E\left(\max_{\substack{1\le i\le m\\ 1\le j\le n}} |S_{ij}|\right)^2 \le c(\log m)^2(\log n)^2 \sum_{k=1}^{m}\sum_{l=1}^{n} E|X_{kl}|^2. \tag{2.6}$$

**Proof.** For $m = 1$ and $n = 1$, the inequality is trivial. If $m > 1$, let $s$ be an integer such that $2^{s-1} < m \le 2^s$. And if $n > 1$, let $t$ be an integer such that $2^{t-1} < n \le 2^t$. We can assume that $m, n > 1$. We assign $X_{ij}$ to the point $(i, j)$ of integer in $(0, 2^s]\times(0, 2^t]$ (if $m < i \le 2^s$ or $n < j \le 2^t$, set $X_{ij} = 0$). Divide the interval $(0, 2^s]$ into $(0, 2^{s-1}]$ and $(2^{s-1}, 2^s]$, each of these two intervals into two halves, and so on. Then the elements of the $i$th partition are of length $2^{s-i}$, $i = 0, \dots, s$. Also, divide the interval $(0, 2^t]$ in the same way. Then we obtain the $(i, j)$th partition $P_{ij}$ of $(0, 2^s]\times(0, 2^t]$ by the $i$th partition of $(0, 2^s]$ and the $j$th partition of $(0, 2^t]$. Every rectangle $(0, i]\times(0, j]$ is the sum of at most $(s+1)(t+1)$ disjoint subrectangles each of which belongs to a different partition. We can write $S_{ij} = \sum_{k=0}^{s}\sum_{l=0}^{t} Y_{kl;ij}$, where $Y_{kl;ij}$ is the sum of all r.v.'s belonging to the rectangle $(a, b]\times(c, d]$, $b - a = 2^k$ and $d - c = 2^l$, which may or may not be a summand of $(0, i]\times(0, j]$ so that some $Y_{kl;ij}$ may vanish. Let $T_{ij} = \sum_{k=1}^{2^i}\sum_{l=1}^{2^j}|Y_{kl}|^2$, where $Y_{kl}$ is the sum of all r.v.'s which belong to the $(k, l)$-element of $P_{ij}$. If we put $T = \sum_{i=0}^{s}\sum_{j=0}^{t} T_{ij}$, by the elementary Schwarz inequality, we obtain

$$|S_{ij}|^2 \le (s+1)(t+1)\sum_{k=0}^{s}\sum_{l=0}^{t}|Y_{kl;ij}|^2 \le (s+1)(t+1)T. \tag{2.7}$$

Since $ET_{ij} \le \sum_{k=1}^{m}\sum_{l=1}^{n} E|X_{kl}|^2$, $ET \le (s+1)(t+1)\sum_{k=1}^{m}\sum_{l=1}^{n} E|X_{kl}|^2$. It follows that

$$\begin{aligned} E\left(\max_{\substack{1\le i\le m\\ 1\le j\le n}} |S_{ij}|^2\right) &\le (s+1)^2(t+1)^2\sum_{k=1}^{m}\sum_{l=1}^{n} E|X_{kl}|^2 \\ &\le c(\log m)^2(\log n)^2\sum_{k=1}^{m}\sum_{l=1}^{n} E|X_{kl}|^2. \end{aligned} \tag{2.8}$$

□

**Theorem 2.3.** *Let $\{X_{ij}\}$ be a double sequence of pairwise independent random variables. If $P\{|X_{mn}| \ge t\} \le P\{|X| \ge t\}$ for all nonnegative real numbers $t$ and $E|X|^p(\log^+|X|)^3 < \infty$, for $1 < p < 2$, then*

$$\lim_{m\vee n\to\infty}\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - EX_{ij})}{(mn)^{1/p}} = 0 \quad \textit{a.s.} \tag{2.9}$$

**Proof.** We denote by $S_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ij}$, $S'_{mn}=\sum_{i=1}^{m}\sum_{j=1}^{n}X'_{ij}$. Then we obtain the inequalities

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P\{X_{ij}\neq X'_{ij}\}&=\sum_{k=1}^{\infty}d_kP\{|X_{11}|>k^{1/p}\}\\&\le\sum_{k=1}^{\infty}d_kP\{|X|>k^{1/p}\}\\&=\sum_{i=1}^{\infty}\left(\sum_{k=1}^{i}d_k\right)\int_{i^{1/p}}^{(i+1)^{1/p}}dF(x)\\&\le c\sum_{i=1}^{\infty}i\log i\int_{i^{1/p}}^{(i+1)^{1/p}}dF(x)\\&\le cE|X|^p\log^+|X|<\infty,\end{aligned}\tag{2.10}$$

Hence, by the Borel-Cantelli lemma,

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-X'_{ij})}{(mn)^{1/p}}\longrightarrow 0\quad\text{a.s.}\tag{2.11}$$

Now, we use Chebyshev's inequality and Lemma 2.1 to obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}P\left\{\left|\frac{S'_{2^k2^l}-ES'_{2^k2^l}}{(2^k2^l)^{1/p}}\right|>\epsilon\right\}&\le c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\operatorname{Var}S'_{2^k2^l}}{(2^k2^l)^{2/p}}\\&=c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{(2^k2^l)^{2/p}}\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}\operatorname{Var}X'_{ij}\\&\le c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{EX'^{\,2}_{ij}}{(ij)^{2/p}}\\&\le cE|X|^p\log^+|X|^p<\infty,\end{aligned}\tag{2.12}$$

which follows easily by summation by parts. It follows that

$$\frac{S'_{2^k2^l}-ES'_{2^k2^l}}{(2^k2^l)^{1/p}}\longrightarrow 0\quad\text{a.s.}\tag{2.13}$$

And let

$$\begin{aligned}T_{kl}&=\max_{\substack{2^k\le m<2^{k+1}\\2^l\le n<2^{l+1}}}\left|\frac{S^*_{2^k2^l}}{(2^k2^l)^{1/p}}-\frac{S^*_{mn}}{(mn)^{1/p}}\right|\\&\le\frac{|S^*_{2^k2^l}|}{(2^k2^l)^{1/p}}+\max_{\substack{2^k\le m<2^{k+1}\\2^l\le n<2^{l+1}}}\frac{|S^*_{mn}|}{(mn)^{1/p}},\end{aligned}\tag{2.14}$$

where $S^*_{mn}=S'_{mn}-ES'_{mn}$.

By using Lemma 2.2, we obtain, for any $\epsilon>0$,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}P\left[\max_{\substack{2^k\le m\le 2^{k+1}\\ 2^l\le n\le 2^{l+1}}}\frac{|S^*_{mn}|}{(mn)^{1/p}}\ge\frac{\epsilon}{2}\right]&\le c\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{(2^k2^l)^{2/p}}E\left(\max_{\substack{2^k\le m\le 2^{k+1}\\ 2^l\le n\le 2^{l+1}}}|S^*_{mn}|\right)^2\\&\le c\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(k+1)^2(l+1)^2}{(2^{k+1}2^{l+1})^{2/p}}\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}}E|X'_{ij}|^2\\&\le c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(\log_2 ij)^2}{(ij)^{2/p}}E|X'_{ij}|^2,\end{aligned}\tag{2.15}$$

where the last inequality follows easily be summation by parts. But

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(\log_2 ij)^2}{(ij)^{2/p}}E|X'_{ij}|^2&\le\sum_{k=1}^{\infty}\frac{d_k(\log_2 k)^2}{k^{2/p}}\int_0^{k^{1/p}}x^2\,dF(x)\\&\le\sum_{i=0}^{\infty}\left(\sum_{k=i+1}^{\infty}\frac{d_k(\log_2 k)^2}{k^{2/p}}\right)\int_{i^{1/p}}^{(i+1)^{1/p}}x^2\,dF(x)\\&\le c\sum_{i=0}^{\infty}i^{1-(2/p)}(\log i)^3\int_{i^{1/p}}^{(i+1)^{1/p}}x^2\,dF(x)\\&\le cE|X|^p\left(\log^+|X|^p\right)^3<\infty,\end{aligned}\tag{2.16}$$

where we use $\sum_{k=1}^{\infty}\frac{d_k(\log_2 k)^2}{k^{2/p}}=O\left(\frac{(\log i)^3}{i^{(2/p)-1}}\right)$ which follows by summation by parts. Hence, (2.13), (2.15), and (2.16) give us

$$\frac{S'_{mn}-ES'_{mn}}{(mn)^{1/p}}\longrightarrow 0\quad\text{a.s.}\tag{2.17}$$

Combining (2.11) and (2.17), we get

$$\frac{S_{mn}-ES'_{mn}}{(mn)^{1/p}}\longrightarrow 0\quad\text{a.s.}\tag{2.18}$$

Since

$$\frac{S_{mn}-ES_{mn}}{(mn)^{1/p}}=\frac{S_{mn}-ES'_{mn}}{(mn)^{1/p}}-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}E|X''_{ij}|}{(mn)^{1/p}},\tag{2.19}$$

it remains to prove that the second term of the right-hand side converges to 0 a.s. By Lemma 2.1(b), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E|X''_{ij}|}{(2^k2^l)^{1/p}}&\le c\sum_{i,j=1}^{\infty}\frac{E|X''_{ij}|}{(ij)^{1/p}}\\&\le cE|X|^p\log^+|x|<\infty,\end{aligned}\tag{2.20}$$

from which, it follows that

$$\lim_{k\vee l\longrightarrow\infty}\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E|X''_{ij}|}{(2^k2^l)^{1/p}}=0.\tag{2.21}$$

But since

$$\begin{aligned} T'_{kl} &= \max_{\substack{2^k \le m \le 2^{k+1} \\ 2^l \le n \le 2^{l+1}}} \left| \frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)^{1/p}} - \frac{\sum_{i=1}^{2^k} \sum_{j=1}^{2^l} E|X''_{ij}|}{(2^k 2^l)^{1/p}} \right| \\ &\le \frac{c}{(2^{k+1}2^{l+1})^{1/p}} \sum_{i=1}^{2^{k+1}} \sum_{j=1}^{2^{l+1}} E|X''_{ij}|, \end{aligned} \tag{2.22}$$

$T'_{kl}$ converges to 0 which implies that, by (2.21),

$$\frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)^{1/p}} \longrightarrow 0. \tag{2.23}$$

This completes the proof. □

**Corollary 2.4.** *Let $\{X_{ij}\}$ be a double sequence of pairwise i.i.d. random variables with $E|X_{11}|^p(\log^+|X_{11}|)^3 < \infty$, for $1 < p < 2$. Then*

$$\lim_{m \vee n \to \infty} \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij})}{(mn)^{1/p}} = 0 \quad \text{a.s.} \tag{2.24}$$

**Remark.** The generalization to $r$-dimensional arrays of random variables can be obtained easily under the condition $E|X|^p(\log^+|X|)^{r+1} < \infty$.

**Theorem 2.5.** *Let $\{X_{ij}\}$ be a double sequence of pairwise independent random variables. If $P\{|X_{ij}| \ge t\} \le P\{|X| \ge t\}$ for all nonnegative real numbers $t$ and $E|X|^p \log^+ |X| < \infty$, $1 < p < 2$, then*

$$\frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij})}{(mn)^{1/p}} \longrightarrow 0 \quad \text{in } L^1 \text{ as } m \vee n \longrightarrow \infty. \tag{2.25}$$

**Proof.** Since $\{X_{ij}\}$ are pairwise independent, $\{X'_{ij} - EX'_{ij}\}$ are orthogonal which implies that

$$E\left| \frac{\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - EX'_{ij})}{(mn)^{1/p}} \right|^2 \le \frac{\sum_{i=1}^m \sum_{j=1}^n E|X'_{ij}|^2}{(mn)^{2/p}}. \tag{2.26}$$

Since

$$\begin{aligned} E\left| \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij})}{(mn)^{1/p}} \right| &\le E\left| \frac{\sum_{i=1}^m \sum_{j=1}^n (X'_{ij} - EX'_{ij})}{(mn)^{1/p}} \right| \\ &\quad + 2\frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)^{1/p}}, \end{aligned} \tag{2.27}$$

it suffices to show that $(\sum_{i=1}^m \sum_{j=1}^n E|X'_{ij}|^2)/(mn)^{2/p}$ converges to 0 as $m \vee n \longrightarrow 0$. But this can be shown by a method similar to that used in the proof of (2.23) in Theorem 2.3. □

**Corollary 2.6.** *Let $\{X_{ij}\}$ be a double sequence of pairwise i.i.d. random variable with $E|X_{11}|^p \log^+ |X_{11}| < \infty$, for $1 < p < 2$. Then*

$$\frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - EX_{ij})}{(mn)^{1/p}} \longrightarrow 0 \quad \text{in } L^1 \text{ as } m \vee n \longrightarrow \infty. \tag{2.28}$$

**Remark.** The generalization to $r$-dimensional arrays of random variables can be obtained under the condition $E|X|^p(\log^+|X|)^{r+1} < \infty$.

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