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# SYMMETRIC AND PERMUTATIONAL GENERATING SET OF THE GROUPS $A_{kn+1}$ AND $S_{kn+1}$ USING $S_n$ AND AN ELEMENT OF ORDER $k$

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**ABSTRACT.** In this paper we will show how to generate $A_{kn+1}$ and $S_{kn+1}$ using a copy of $S_n$ and an element of order $k$ in $A_{kn+1}$ and $S_{kn+1}$ respectively, for all positive integers $n \geq 2$ and all positive integers $k \geq 2$. We will also show how to generate $A_{kn+1}$ and $S_{kn+1}$ symmetrically using $n$ elements each of order $k$, for all $n \geq 2$ and all even integers $k \geq 2$.

**KEY WORDS AND PHRASES:** Symmetric generators, Group presentation, Doubly transitive groups.
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## 1. INTRODUCTION

Hammas [1] showed that $A_{2n+1}$ can be presented as

$$G = A_{2n+1} = \langle X, Y, T \mid \langle X, Y\rangle = S_n, T^2 = [T, S_{n-1}] = 1 \rangle$$

for $n = 4, 6$, where $[T, S_{n-1}]$ means that $T$ commutes with $Y$ and with $X^{-2}YX$, (the generators of $S_{n-1}$). The relations of the symmetric group $S_n = \langle X, Y\rangle$ of degree $n$ are found in Coxeter and Moser [2]. Some relations must be added to the presentation that generates $A_{2n+1}$ in order to complete the coset enumeration. Also Hammas [1] showed that, for $n = 4, 6$, the group $A_{2n+1}$ can be symmetrically generated by $n$ elements each of order 2 and of the form $T_0, T_1, ..., T_{n-1}$, where $T_i = T^{X^i} = X^{-i}TX^i$ and $T, X$ satisfy the relations of the group $A_{2n+1}$. The set $\{T_0, T_1, ..., T_{n-1}\}$ is called the symmetric generating set of $A_{2n+1}$ ( see the Definition 2.1 in Section 2 ).

Hammas [3] showed that $A_{2n+1}$ can be presented as

$$A_{2n+1} = \langle X,Y,T \mid \langle X,Y\rangle = S_n, T^2 = [T,Y] = [T,X^{-2}YX] = (XT)^{2n+1} = (YT_{n-2})^{10}\rangle$$

when $n$ is an even integer and $S_{2n+1}$ can be presented as

$$S_{2n+1} = \langle X,Y,T \mid \langle X,Y\rangle = S_n, T^2 = [T,Y] = [T,X^{-2}YX] = (XT)^{n(n+1)} = (YT_{n-2})^{10}\rangle.$$

when $n$ is odd. Note that the order of the third generator, $T$, was always 2.

Also, it has been shown by Hammas [3] that for all $n \geq 2$ the groups $A_{2n+1}$ and $S_{2n+1}$ can be symmetrically generated using $n$ elements each of order 2, and of the form $T_0, T_1, ..., T_{n-1}$, where $T_i = T^{X^i} = X^{-i}TX^i$ and $T, X$ satisfy the relations of the groups $A_{2n+1}$. and $S_{2n+1}$.

In this paper, we give a generalization of the results obtained by Hammas [1-3]. We will show that, for all $k \geq 2$ and for all $n \geq 2$, the group generated by $X$, $Y$ and $T$ is the alternating group $A_{kn+1}$ when $n$ and $k$ are all even integers and is the symmetric group $S_{kn+1}$ otherwise. Moreover, relations will be given to show that, for all $k \geq 2$ and for all $n \geq 2$, the group

$$G = < X, Y, T \mid < X, Y > = S_n,\ T^k = [T, S_{n-1}] = 1 >$$

is $A_{kn+1}$ when $n$ and $k$ are both even and $S_{kn+1}$ otherwise. We give permutations that generate $A_{kn+1}$ and $S_{kn+1}$ which satisfy the conditions given in the presentation of the group $G$. Further, we prove that, when $k$ is an even integer, $G$ can be symmetrically generated by $n$ permutations each of order $k$ of the form $T_0, T_1, ..., T_{n-1}$, where $T_i = T^{X^i} = X^{-i}TX^i$, satisfying the condition hat $T_0$ commutes with the generators of the group $S_{n-1}$.

## 2. PRELIMINARY RESULTS

**THEOREM 2.1.** Let $1 \leq a \neq b \leq n$ be any integers. Let $G$ be the group generated by the $n$-cycle $(1, 2, ..., n)$ and the 3-cycle $(n, a, b)$ where the highest common factor **hcf**$(n, a, b) = 1$. If $n$ is an odd integer then $G = A_n$ while, if $n$ is even, then $G = S_n$.

**DEFINITION 2.1.** Let $G$ be a group and $\Gamma = \{ T_0, T_1, ..., T_{n-1} \}$ be a subset of $G$ where $T_i = T^{X^i} = X^{-i}TX^i$ for all $i = 0, 1, ..., n-1$. Let $S_n$ - a copy of the symmetric group of degree $n$ - be the normalizer in $G$ of the set $\Gamma$. We define $\Gamma$ to be a symmetric generating set of $G$ if and only if $G = <\Gamma>$ and $S_n$ permutes $\Gamma$ doubly transitively by conjugation, i.e., $\Gamma$ is realizable as an inner automorphism.

## 3. PERMUTATIONAL GENERATING SET OF $A_{kn+1}$ AND $S_{kn+1}$

**THEOREM 3.1.** For all $n \geq 2$ and all $k \geq 2$, $A_{kn+1}$ can be generated using a copy of $S_n$ and an element of order $k$ in $A_{kn+1}$ when $n$ and $k$ are both even and $S_{kn+1}$ can be generated using a copy of $S_n$ and an element of order $k$ in $S_{kn+1}$ if $n$ or $k$ is odd.

**PROOF.** Let $X = (1,2,...,n)(n+1,n+2,...,2n)...((k-1)n+1,(k-1)n+2,...,kn)$, $Y = (n-1,n)...(kn-1,kn)$ and $T = (1, n+1, 2n+1, 3n+1,..., (k-2)n+1, kn+1)(2, n+2, 2n+2,..., (k-1)n+2)...(n, 2n,..., kn)$ be three permutations; the first of order $n$, the second of order 2 and the third of order $k$. Let $H$ be the group generated by $X$ and $Y$. By a result of Burnside and Moore, ( see Coxeter and Moser[2] ), the group $H$ is the symmetric group $S_n$. Let $G$ be the group generated by $X$, $Y$ and $T$. We have two cases :

Case 1 Let $k$ be an odd integer. Let $\alpha = [X, T]$. Then $\alpha = ( 1, (k-1)n+1, kn+1, (k-1)n+2, 2 )$. Let $\beta = \alpha^3\alpha^T$. Then

$$\beta = ( 1, (k-1)n+2 )( 2, kn+1 )( n+1, (k-1)n+1, n+2 ).$$

Let $\delta = \alpha X \beta^{T^0} \beta^{T^2} ... \beta^{T^{k-3}} \alpha Y^X$. Hence

$$\delta = ( 1, kn, 3n, n+2, ..., 2n-1, n+1, 2n+2, ..., 3n-1, 2n+1, 2n, 5n, 3n+2, ..., 4n-1, 3n+1, 4n+2, ..., 5n-1, 4n+1, 4n, 7n, 5n+2, ..., 6n-1, 5n+1, 6n+2, ..., 7n-1, 6n+1, 6n, 9n, ..., (k-6)n+2, ..., (k-5)n-1, (k-6)n+1, (k-5)n+2, ..., (k-4)n-1, (k-5)n+1, (k-5)n, (k-2)n, (k-4)n+2, ..., (k-3)n-1, (k-4)n+1, (k-3)n+2, ..., (k-2)n-1, (k-3)n+1, (k-3)n, (k-2)n+2, ..., (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, ..., kn-1, (k-1)n+1, n, 2, (k-1)n+2, 3, ..., n-1 )$$

which is a cycle of length $kn+1$. Let $K = <\delta, \beta^2>$. We claim that $K$ is either $A_{kn+1}$ or $S_{kn+1}$. To show this, let $\theta$ be the mapping which takes the element in the position $i$ of the permutation $\beta$ into the element $i$ in the permutation $(1, 2, ..., kn+1)$. Under the mapping $\theta$, the group $K$ will be mapped into the group

$$\theta(K) = <(1, 2, ..., kn+1), (n+1, 4, (k-1)n+1)>.$$

Since $k$ is an odd integer the highest common factor **hcf**$(n+1, 4, (k-1)n+1) = 1$. Hence by Theorem 2.1, if $n$ is an odd integer then $\theta(K)$ is $S_{kn+1}$. Hence $G$ is $S_{kn+1}$. But if $n$ is an even integer then $\theta(K)$ is $A_{kn+1}$. Since $k$ is an odd integer, $Y$ is an odd permutation. The action of the generators of $A_{kn+1}$ on $Y$ is not trivial and therefore $G$ is the symmetric group $S_{kn+1}$.

Case 2 Let $k$ be an even integer. Let $\alpha = [X, T]$. Then $\alpha = (1, (k-1)n+1, kn+1, (k-1)n+2, 2)$. Let $\beta = \alpha^3\alpha^T$. Then

$$\beta = (1, (k-1)n+2)(2, kn+1)(n+1, (k-1)n+1, n+2).$$

Let $\delta = \alpha X \beta^T \beta^{T^3} \dots \beta^{T^{(k-4)/2}} \alpha Y^X$. Hence

$$\delta = (1, 2, 2n, n, n+2, ..., 2n-1, n+1, kn, 4n, 2n+2, ..., 3n-1, 2n+1, 3n+2, ...,$$
$$4n-1, 3n+1, 3n, 6n, 4n+2, ..., 5n-1, 4n+1, 5n+2, ..., 6n-1, 5n+1, 5n, 8n, ...,$$
$$(k-6)n+2, ..., (k-5)n-1, (k-6)n+1, (k-5)n+2, ..., (k-4)n-1, (k-5)n+1, (k-5)n,$$
$$(k-2)n, (k-4)n+2, ..., (k-3)n-1, (k-4)n+1, (k-3)n+2, ..., k-2)n-1, (k-3)n+1,$$
$$(k-3)n, (k-2)n+2, ..., (k-1)n-1, (k-2)n+1, (k-1)n, kn+1, (k-1)n+3, ...,$$
$$kn-1, (k-1)n+1, (k-1)n+2, 3, ..., n-1)$$

which is a cycle of length $kn+1$. Let $K = <\delta, \beta^2>$. Using the same method used above we can easily show that $K$ is the alternating group $A_{kn+1}$. Now, since $k$ is an even integer, then, if $n$ is an even integer too, $G$ has to be the alternating group $A_{kn+1}$ or a proper subgroup of it. Since $K$ is the alternating group $A_{kn+1}$ then $G$ is the alternating group $A_{kn+1}$. But if $n$ is an odd integer then $T$, the third generator of $G$, is an odd permutation. Since the action of the generators of the group $K$ on the element $T$ is not trivial, the group $<\delta, \beta^2, T>$ is the symmetric group $S_{kn+1}$. Hence $G$ is the symmetric group $S_{kn+1}$.

## 4. SYMMETRIC PERMUTATIONAL GENERATING SET OF $A_{kn+1}$ and $S_{kn+1}$

**THEOREM 4.1.** Let $X$, $Y$ and $T$ be the permutations described in Theorem 3.1 where $T^k = 1$. Let $\Gamma = \{T_0, T_1, ..., T_{n-1}\}$, where $T_i = T^{X^i}$. Let $k$ be an even integer. If $n$ is an even integer too, then the set $\Gamma$ generates the alternating group $A_{kn+1}$ symmetrically, while, if $n$ is an odd integer, then the set $\Gamma$ generates the symmetric group $S_{kn+1}$ symmetrically.

**PROOF.** Let $T_0 = (1, n+1, 2n+1, 3n+1, ..., kn+1)(2, n+2, 2n+2, ..., (k-1)n+2)...(n, 2n, ..., kn)$, $T_1 = T^X = (1, n+1, 2n+1, ..., (k-1)n+1)(2, n+2, 2n+2, ..., kn+1)...(n, 2n, ..., kn), ..., T_{n-1} = T^{X^{n-1}} = (1, n+1, 2n+1, ..., (k-1)n+1)(2, n+2, 2n+2, ..., (k-1)n+2)...(n, 2n, ..., kn+1)$. Let $H = <\Gamma>$. We claim that $H \equiv A_{kn+1}$ or $S_{kn+1}$. To show this, suppose first that $n$ is an odd integer. Let $\ell = \frac{k}{2}$ if $\frac{k}{2}$ is even and $\ell = \frac{k}{2}+1$ if $\frac{k}{2}$ is odd and let $r = \ell - 1$ if $\frac{k}{2}$ is odd and $r = \ell + 1$ if $\frac{k}{2}$ is even. Consider the element $\alpha = (T_0^{(T_1 T_2 \dots T_{n-2})})T_{n-1}$. We find that

$$\alpha = (1, 2n+1, 4n+1, ..., \ell n+1, (\ell+1)n+1, (\ell+3)n+1, ..., (k-1)n+1, n+1, 3n+1, 5n+1, ...,$$
$$rn+1, (r+1)n+2, (r+3)n+2, ..., (k-1)n+2, n+2, 3n+2, 5n+2, ..., rn+2, (r+3)n+2, (r+5)n+2, ...,$$

$(k\text{-}2)n+2, 2, 2n+2, 4n+2, ..., (\ell\text{-}2)n+2, \ell n+3, (\ell+2)n+3, ..., (k\text{-}2)n+3, 3, 2n+3, 4n+3, ...,$

$(\ell\text{-}2)n+3, (\ell+1)n+3, (\ell+3)n+3, ..., (k\text{-}1)n+3, n+3, 3n+3, 5n+3, ..., (r\text{-}2)n+3, rn+4, (r+2)n+4, ...,$

$(k\text{-}1)n+4, n+4, 3n+4, 5n+4, ..., (r\text{-}2)n+4, (r+1)n+4, (r+3)n+4, ..., (k\text{-}2)n+4, 4, 2n+4, 4n+4, ...,$

$(\ell\text{-}4)n+4, (\ell\text{-}2)n+5, \ell n+5, (\ell+2)n+5, ..., (k\text{-}2)n+5, 5, 2n+5, 4n+5, ..., (\ell\text{-}4)n+5, (\ell\text{-}1)n+5,$

$(\ell+1)n+5, (\ell+3)n+5, ..., (k\text{-}1)n+5, n+5, 3n+5, 5n+5, ..., (r\text{-}4)n+5, (r\text{-}2)n+6, rn+6, (r+2)n+6, ...,$

$(k\text{-}1)n+6, n+6, 5n+6, 7n+6, ..., (r\text{-}4)n+6, (r\text{-}1)n+6, (r+1)n+6, (r+3)n+6, ..., (k\text{-}2)n+6, 6, 2n+6,$

$4n+6, ..., (\ell\text{-}6)n+6, (\ell\text{-}4)n+7, (\ell\text{-}2)n+7, \ell n+7, (\ell+2)n+7, ..., (k\text{-}2)n+7, 7, 2n+7, 4n+7, ...,$

$(\ell\text{-}6)n+7, (\ell\text{-}3)n+7, (\ell\text{-}1)n+7, (\ell+1)n+7, ..., (k\text{-}1)n+7, n+7, 3n+7, 5n+7, ..., (r\text{-}6)n+7,$

$(r\text{-}4)n+8, (r\text{-}2)n+8, rn+8, ..., (k\text{-}1)n+8, n+8, 5n+8, 7n+8, ..., (r\text{-}6)n+8, (r\text{-}3)n+8, (r\text{-}1)n+8,$

$(r+1)n+8, (r+3)n+8, ..., (k\text{-}2)n+8, ..., n\text{-}2, ..., n, 3n, ...(k\text{-}1)n, kn, 2n, 4n, ..., (k\text{-}2)n, kn+1, n\text{-}1,$

$2n\text{-}2, 3n\text{-}3, ..., (n\text{-}2)n\text{-}(n\text{-}2), (n\text{-}1)n+1, (n+1)n+1, (n+3)n+1, ..., (k\text{-}2)n+1)$

which is a cycle of length $kn+1$. Let $\beta = T^{-1}T^X$. Therefore $\beta = (1,2,(k\text{-}1)n+2,kn+1,(k\text{-}1)n+1)$ which is a cycle of length 5. Let $\gamma = \beta^3\beta^T$. Since $\gamma = (2,n+2,n+1)$ then using the same method used in Theorem 3.1 above we get $H_1 = <\alpha, \gamma, T> \cong S_{kn+1}$. Hence $H \cong H_1 \cong \theta(H_1) \cong S_{kn+1}$. In the same way we can show that, when $n$ is an even integer, $H \cong A_{kn+1}$.

The above results can be summarised in the following table:

| | **n** | **k** | $G = <X, Y, T>$ | $<X, T>$ | $<\Gamma>$ |
|---|---|---|---|---|---|
| **1** | **even** | **even** | $A_{kn+1}$ | $A_{kn+1}$ | $A_{kn+1}$ |
| **2** | **even** | **odd** | $S_{kn+1}$ | $S_{kn+1}$ | $A_{kn+1}$ |
| **3** | **odd** | **even** | $S_{kn+1}$ | $S_{kn+1}$ | $S_{kn+1}$ |
| **4** | **odd** | **odd** | $S_{kn+1}$ | $A_{kn+1}$ | $A_{kn+1}$ |

where

$$G = <X,Y,T \mid <X,Y> = S_n, T^k = [T,Y] = [T,X^{-2}YX] = ([X,T]X)^n = (YX)^{n-1}, [X,T]^5 =$$
$$(T^X T^{-1})^5 = (TT^{([X,T]^2[X,T]^T)}T^{-1})^k = (T^{([X,T]^2[X,T]^T)}T^{-1})^6 = (T[X,T]^X)^r = 1 >$$

where $r = k(k\text{-}1)$ when $k$ is odd and $r = 2k(k+1)$ when $k$ is even for all $n, k \geq 3$.

From the above table we can see that in the case when $k$ is an odd integer the set $\Gamma = \{T_0, T_1, .., T_{n-1}\}$ cannot generate the symmetric group $S_{kn+1}$ symmetrically. As a matter of fact, as we verified using the GAP package, the set $\Gamma$ generates the alternating group $A_{kn+1}$ symmetrically. But unfortunately we haven't found a hand proof of this case yet.

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