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ON SOME COMPACT ALMOST KÄHLER LOCALLY SYMMETRIC SPACE

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ABSTRACT. In the framework of studying the integrability of almost Kähler manifolds, we prove that if a compact almost Kähler locally symmetric space M is a weakly $*$-Einstein manifold with non-negative $*$-scalar curvature, then M is a Kähler manifold.

KEY WORDS AND PHRASES: Almost Kähler manifolds, locally symmetric spaces and weakly $*$-Einstein manifolds.
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1. INTRODUCTION

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the corresponding Kähler form is closed, or equivalently $\underset{X,Y,X}{\mathfrak{S}} g((\nabla_X J)Y, Z) = 0$ for all X, Y, $Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M and $\underset{X,Y,Z}{\mathfrak{S}}$ denotes the cyclic sum with respect to X, Y, Z. By the definition, a Kähler manifold ($\nabla J = 0$) is necessarily an almost Kähler manifold. It is well-known that if the almost complex structure of an almost Kähler manifold M is integrable, then M is a Kähler manifold. A non-Kähler, almost Kähler manifold is called a strictly almost Kähler manifold. Several examples of strictly almost Kähler manifolds have been constructed ([1], [6], [9], [13] and so on).

Concerning the integrability of almost Kähler manifolds, the following conjecture by Goldberg is known ([4]).

CONJECTURE. The almost complex structure of a compact almost Kähler Einstein manifold is integrable.

About the above conjecture, some progress have been made under some additional curvature conditions. For example, Sekigawa proved that the conjecture is true if the scalar curvature is non-negative ([11]). Further result have been obtained ([2], [3], [7], [8], [10], [12] and so on).

On any almost Hermitian manifold, we can define Ricci $*$-tensor, an analogue of the Ricci tensor, but involving the almost complex structure (see (2.1) below for the definition). On a Kähler manifold, the Ricci tensor and the Ricci $*$-tensor coincide. Therefore, it is natural to consider star-version of the Goldberg conjecture. An almost Hermitian manifold is called weakly $*$-Einstein if the Ricci $*$-tensor is a (not necessarily constant) multiple of the metric and $*$-Einstein if the Ricci $*$-tensor is a constant multiple of the metric.

In the present paper, concerning the star-version of the Goldberg conjecture, we shall prove the following.

THEOREM. Let $M = (M, J, g)$ be a compact almost Kähler locally symmetric space which is a weakly $*$-Einstein manifold with non-negative $*$-scalar curvature. Then, M is a Kähler manifold.

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2. PRELIMINARIES

Let $M = (M, J, g)$ be a $2n$-dimensional almost Hermitian manifold with the almost Hermitian structure (J, g) and Ω the corresponding Kähler form of M defied by $\Omega(X,Y) = g(X, JY)$ for X, $Y \in \mathfrak{X}(M)$. We assume that M is oriented by the volume form $dM = \frac{(-1)^n}{n!}\Omega^n$. Let ∇ be the Riemannian connection and R its curvature tensor given by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ for X, Y, $Z \in \mathfrak{X}(M)$. We denote by ρ and τ the associated Ricci tensor and the scalar curvature, respectively. Moreover, let ρ^*, respectively τ^*, denote the Ricci $*$-tensor, respectively the $*$-scalar curvature, defined by

$$\rho^*(x,y) = g(Q^*x, y) = \text{ trace } \left(z \mapsto R(x, Jz)Jy\right), \tag{2.1}$$
$$\tau^* = \text{ trace } Q,$$

for x, y, $z \in T_pM$, the tangent space of M at $p \in M$. By using the first Bianchi identity, we have easily

$$\rho^*(x,y) = \frac{1}{2} \text{ trace } \left(z \mapsto R(x, Jy)Jz\right). \tag{2.2}$$

Thus, in general, ρ^* is neither symmetric nor skew-symmetric. But it satisfies the following identify.

$$\rho^*(X,Y) = \rho^*(JY, JX),$$

for any X, $Y \in \mathfrak{X}(M)$.

Let $\{e_1, \dots, e_{2n}\}$ be an orthonormal basis of T_pM at any point $p \in M$. In the present paper, we shall adopt the following notational convention:

$$R_{ijkl} = g(R(e_i, e_j)e_k, e_l),\ R_{\bar{i}jkl} = g(R(Je_i, e_j)e_k.e_l), \cdots, R_{\bar{i}\bar{j}\bar{k}\bar{l}} = g(R(Je_i, Je_j)Je_k, Je_l),$$
$$\rho_{ij} = \rho(e_i, e_j), \dots, \rho_{\bar{i}\bar{j}} = \rho(Je_i, Je_j), \quad \rho^*_{ij} = \rho^*(e_i, e_j), \dots, \rho^*_{\bar{i}\bar{j}} = \rho^*(Je_i, Je_j),$$
$$J_{ij} = g(Je_i, e_j), \quad \nabla_i J_{jk} = g((\nabla_{e_i} J)e_j, e_k),$$

and so on, where the latin indices run over the range 1, 2, ... , $2n$. Then, we have easily

$$J_{ij} = -J_{ji}, \qquad \nabla_i J_{jk} = -\nabla_i J_{kj}, \qquad \nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}. \tag{2.3}$$

Now, we shall define smooth functions A, B on M respectively by

$$A = \sum_{a,i,j,k,l=1}^{2n} (\nabla_a J_{ji})(\nabla_a J_{kl})R_{ijkl},$$
$$B = \sum_{a,b,i,j,k,l=1}^{2n} (\nabla_a J_{ij})(\nabla_a J_{kl})(\nabla_b J_{ij})\nabla_b J_{kl},$$

at any point $p \in M$.

Now, we assume that $M = (M, J, g)$ is an almost Kähler manifold. Then it is known that M is a quasi-Kähler manifold, namely, the equality

$$\nabla_i J_{jk} = -\nabla_{\bar{i}} J_{\bar{j}k} \tag{2.4}$$

is valid. Thus, it follows immediately that M is a semi-Kähler manifold, namely, the equality

$$\sum_{a=1}^{2n} \nabla_a J_{ai} = 0 \tag{2.5}$$

is valid. We now recall the following curvature identity established by Gray ([5]):

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}k\bar{l}} + R_{i\bar{j}\bar{k}l} = 2\sum_{a=1}^{2n}(\nabla_a J_{ij})\nabla_a J_{kl}. \tag{2.6}$$

From (2.1) $\sim$ (2.3) and (2.6), we have easily

$$\rho^*_{ij} + \rho^*_{ji} - \rho_{ij} - \rho_{\bar{i}\bar{j}} = \sum_{a,k=1}^{2n} (\nabla_a J_{ik})\nabla_a J_{jk},$$

and further

$$\|\nabla J\|^2 = 2(\tau^* - \tau). \tag{2.7}$$

On one hand, transvecting $\sum_b (\nabla_b J_{ij})\nabla_b J_{kl}$ with (2.6) and taking account of (2.4), we have easily

$$B = 4A. \tag{2.8}$$

Further, we may easily observe that M is Kähler if and only if $B = 0$ holds identically on M.

3. PROOF OF THEOREM

Let $M = (M, J, g)$ be a $2n$-dimensional compact almost Kähler locally symmetric space which is weakly $*$-Einstein with $\tau^* \geq 0$. We define a smooth vector field ξ on M by

$$\xi = \sum_{a=1}^{2n}\left(\sum_{i,j,k,l=1}^{2n} R_{ijkl} J_{ij} \nabla_a J_{kl}\right) e_a$$

at each point $p \in M$. Then, taking account of $\nabla R = 0$, the Ricci identity and (2.2), we have

$$\begin{aligned}
\operatorname{div} \xi &= \sum_{a,i,j,k,l=1}^{2n} \nabla_a (R_{ijkl} J_{ij} \nabla_a J_{kl}) \\
&= \sum R_{ijkl} (\nabla_a J_{ij}) \nabla_a J_{kl} + \sum R_{ijkl} J_{ij} \nabla^2_{aa} J_{kl} \\
&= A - \sum R_{ijkl} J_{ij} (\nabla^2_{ak} J_{la} + \nabla^2_{al} J_{ak}) \\
&= A + \sum R_{ijkl} J_{ij} (R_{aklb} J_{ba} + R_{akab} J_{lb} + R_{alab} J_{bk} + R_{alkb} J_{ab}) \\
&= A - \sum R_{ijkl} J_{ij} R_{ablk} J_{ab} - 2 \sum R_{ijkl} J_{ij} J_{lb} \rho_{kb} \\
&= A + 4\sum (\rho^*_{ij})^2 - 4 \sum \rho^*_{ij} \rho_{ij}.
\end{aligned} \tag{3.1}$$

Since M is weakly $*$-Einstein, from (3.1), we have

$$\operatorname{div}\xi = A + \frac{2}{n}\tau^*(\tau^* - \tau).$$

Thus, form (2.7) and (2.8), we have finally

$$\operatorname{div}\xi = \frac{1}{4}B + \frac{1}{n}\tau^*\|\nabla J\|^2.$$

Therefore, by Green's Theorem, we obtain a following integral formula:

$$\int_M (nB + 4\tau^*\|\nabla J\|^2)dM = 0.$$

Since $B \geq 0$ and $\tau^* \geq 0$, it must follow that $B = 0$ holds identically on M, and hence M is a Kähler manifold. This completes the proof of Theorem.

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