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LONGEST CYCLES IN CERTAIN BIPARTITE GRAPHS

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ABSTRACT. Let G be a connected bipartite graph with bipartition (X, Y) such that $|X| \geq |Y|$ (≥ 2), $n = |X|$ and $m = |Y|$ Suppose, for all vertices $x \in X$ and $y \in Y$, $\text{dist}(x,y) = 3$ implies $d(x) + d(y) \geq n + 1$ Then G contains a cycle of length $2m$ In particular, if $m = n$, then G is hamiltonian

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1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard and can be found in [1]. Let $G = (V, E)$ be a graph For each vertex $x \in V$, let $D(x) = \{v \in V : v \text{ is adjacent to } x\}$. Then $d(x) = |D(x)|$ is the degree (valency) of x in G

Let G be a 2-connected graph. Suppose, for all vertices, $x, y \in V$, $\text{dist}(x,y) = 2$ implies $\max\{d(x), d(y)\} \geq |V|/2$ Then it was shown in [2] that G is hamiltonian Some generalizations of this result can be found in [3]

The purpose of this paper is to obtain a similar result for bipartite graphs Let G be a connected bipartite graph with bipartition (X, Y) such that $|X| \geq |Y|$ (≥ 2), $n = |X|$ and $m = |Y|$ If $m \neq n$, then G cannot be hamiltonian However, G may contain cycles Suppose, for all vertices $x \in X$ and $y \in Y$, $\text{dist}(x, y)) = 3$ implies $d(x) + d(y) \geq n + 1$ Then we show that G contains a cycle of length $2m$ (Theorem 7) It is also shown that G is 2-connected (Corollary 8)

As shown by an example in Section 3, the condition "$\text{dist}(x,y) = 3$ implies $d(x) + d(y) \geq n + 1$" cannot be replaced by a weaker condition "$\text{dist}(x,y) = 3$ implies $\max(d(x), d(y)) \geq (n+1)/2$ " Also this condition cannot be replaced by "$\text{dist}(x,y) = 3$ implies $d(x) + d(y) \geq m + 1$," if $m \neq n$

2. RESULTS

In this section, we assume that G is a connected bipartite graph with bipartition (X, Y) such that $|X| \geq |Y|$ (≥ 2) Let $n = |X|$ and $m = |Y|$ We also assume that, for all vertices $x \in X$ and $y \in Y$, dist $(x, y) = 3$ implies $d(x) + d(y) \geq n + 1$

If S is a subgraph of G and v is a vertex of S, let $D_S(v) = \{u \in V(S) : u \text{ is adjacent to } v\}$ and $d_S(v) = |D_S(v)|$ Let $P = \{w_1, w_2, ..., w_{2r}\}$ be a longest path of length $2r$ such that $w_1, w_3, ..., w_{2r-1} \in X$ and $w_2, w_4, ..., w_{2r} \in Y$ A path of even length is called an even path

LEMMA 1. The minimum degree of G is at least two

PROOF. Suppose, on the contrary, there exists a vertex $x \in X$ such that $d(x) = 1$ Since G is connected, it is easy to see that there exists some y in G such that $\text{dist}(x,y) = 3$ and so $d(x) + d(y) \geq n + 1$ Since y is not adjacent to x, $d(y) \leq n - 1$ But then $d(x) + d(y) \leq n$, which is impossible. Hence $d(x) \geq 2$ for all $x \in X$. Similarly, $d(y) \geq 2$ for all $y \in Y$

LEMMA 2. If $d_P(w_1) + d_P(w_{2r}) \geq r + 1$, then the vertices of P form a cycle

PROOF. Suppose this is not true. Then w_1 is not adjacent to w_{2r} For each $w_i \in D_P(w_1)$ (with $i \neq 2$), we have $w_{i-1} \notin D_P(w_{2r})$, for otherwise $(w_1, w_i, w_{i+1}, \ldots, w_{2r}, w_{i-1}, w_{i-2}, \ldots, w_2)$ is a cycle of length $2r$ Since $w_1 \notin D_P(w_{2r})$, we have

$$d_P(w_{2r}) \leq \underset{(\text{take out } w_1)}{(r-1)} - \underset{(\text{take out } i=2)}{(d_P(w_1) - 1)}$$

Hence $d_P(w_1) + d_P(w_{2r}) \leq r$, which is a contradiction.

LEMMA 3. If $d(w_1) + d(w_{2r}) \geq n + 1$, then the vertices of P form a cycle.

PROOF. If w_1 is adjacent to w_{2r}, the lemma is true. Hence we can assume that w_1 is not adjacent to w_{2r} By Lemma 2, it is sufficient to show that $d_P(w_1) + d_P(w_{2r}) \geq r + 1$ Since P is a longest even path, either $d_P(w_1) = d(w_1)$ or $d_P(w_{2r}) = d(w_{2r})$

CASE 1. Assume $d_P(w_1) = d(w_1)$ Then $d(w_{2r}) - d_P(w_{2r}) \leq n - r$ and so $d_P(w_{2r}) \geq d(w_{2r}) - n + r$ Hence $d_P(w_1) + d_P(w_{2r}) \geq d(w_1) + d(w_{2r}) - n + r \geq r + 1$

CASE 2. Assume $d_P(w_{2r}) = d(w_{2r})$ Then $d(w_1) - d_P(w_1) \leq m - r$ and so $d_P(w_1) \geq d(w_1) - m + r$ Hence $d_P(w_1) + d_P(w_{2r}) \geq d(w_1) - m + r + d(w_{2r}) \geq (n+1) - m + r = r + 1 + (n-m) \geq r + 1$. Therefore the lemma is true

In the following lemma, we assume that $d_P(w_1) = d(w_1)$ and w_1 is not adjacent to w_{2r} Since, by Lemma 1, $d_P(w_1) = d(w_1) \geq 2$, there exists some $k (2 < k < 2r)$ such that w_1 is adjacent to w_k and k is largest, this means, if $k' > k$, then $w_{k'}$, is not adjacent to w_1.

LEMMA 4. Suppose that $d_P(w_1) = d(w_1)$ and the vertices of P do not form a cycle Then $d_P(w_{k+1}) + d_P(w_{2r}) \leq r$

PROOF. By Lemmas 2 and 3, $d(w_1) + d(w_{2r}) \leq n$ and $d_P(w_1) + d_P(w_{2r}) \leq r$. Hence $\operatorname{dist}(w_1, w_{2r}) > 3$ and so $k \neq 2r - 2$ Thus $k + 2 \neq 2r$. By the choice of k, w_{k+2} is not adjacent to w_1 and so $\operatorname{dist}(w_{k+2}, w_1) = 3$ which implies $d(w_1) + d(w_{k+2}) \geq n + 1$ Hence by Lemma 3, the vertices of P cannot form a path of length $2r$ with ends w_1 and w_{k+2}.

We claim that, for any $w_i \in D_P(w_{k+1})$, $w_{i-1} \notin D_P(w_{2r})$ In fact, if $i = 2$, then w_{2r} is not adjacent to w_1 If $2 < i < k$ and w_{i-1} is adjacent to w_{2r}, then we have $w_1, w_k, w_{k-1}, \ldots, w_i, w_{k+1}, w_{k+2}, \ldots, w_{2r}, w_{i-1}, w_{i-2}, \ldots, w_2$ This is a cycle of length $2r$, which is impossible If $i = k$, then w_{2r} is not adjacent to w_{k-1}, because $\operatorname{dist}(w_1, w_{2r}) > 3$ If $i = k + 2$, then w_{2r} is not adjacent to w_{k+1}, otherwise $\operatorname{dist}(w_1, w_{2r}) = 3$ If $k + 4 \leq i \leq 2r - 2$ and w_{2r} is adjacent to w_{i-1}, then we have $w_1, w_2, \ldots, w_{k+1}, w_i, w_{i+1}, \ldots, w_{2r}, w_{i-1}, w_{i-2}, \ldots, w_{k+2}$ This is a path of length $2r$ with ends w_1 and w_{k+2}, which is impossible Therefore, for any $w_i \in D_P(w_{k+1})$, $w_{i-1} \notin D_P(w_{2r})$ and so $d_P(w_{2r}) \leq r - d_P(w_{k+1})$. Thus $d_P(w_{k+1}) + d_P(w_{2r}) \leq r$

LEMMA 5. If $d_P(w_1) = d(w_1)$ and $d_P(w_{2r}) = d(w_{2r})$, then the vertices of P form a cycle

PROOF. Suppose, on the contrary, that the vertices of P do not form a cycle Let k be as in Lemma 4. From the proof of Lemma 4, we have $\operatorname{dist}(w_1, w_{2r}) > 3$ and so w_{k+1} is not adjacent to w_{2r} If $\operatorname{dist}(w_{k+1}, w_{2r}) = 3$, then $d(w_{k+1}) + d(w_{2r}) \geq n + 1$ Since $d(w_{2r}) = d_P(w_{2r})$, it follows from the proof of Lemma 3 that $d_P(w_{k+1}) + d_P(w_{2r}) \geq r + 1$. But this contradicts Lemma 4. Thus $\operatorname{dist}(w_{k+1}, w_{2r}) > 3$ If there exists some vertex w_i which is adjacent to both w_{k+2} and w_{2r}, then $\operatorname{dist}(w_{k+1}, w_{2r}) = 3$, which is impossible Hence w_{k+2} and w_{2r} cannot have a common neighbor on P and so $d_P(w_{k+2}) + d_P(w_{2r}) \leq r$ Therefore $d_P(w_{k+2}) \leq r - d_P(w_{2r})$. Since $\operatorname{dist}(w_1, w_{k+2}) = 3$, we have $d(w_1) + d(w_{k+2}) \geq n + 1$ Since $d(w_1) = d_P(w_1)$, by the proof of Lemma 3, $d_P(w_1) + d_P(w_{k+2}) \geq r + 1$. Hence, we have $r - d_P(w_{2r}) \geq d_P(w_{k+2}) \geq r + 1 - d_P(w_1)$ Therefore $d_P(w_1) - d_P(w_{2r}) \geq 1$ and so $d(w_1) > d(w_{2r})$ By replacing w_{2r} with w_1 in the above argument, we can also show that $d(w_{2r}) > d(w_1)$ This is impossible Hence the vertices of P form a cycle.

LEMMA 6. There exists a cycle of length $2r$ in G

PROOF. By Lemma 5, we can assume that, for each path P of length $2r$, either $d(w_1) > d_P(w_1)$ or $d(w_{2r}) > d_P(w_{2r})$, otherwise the lemma is true. Let P be a path of length $2r$ with $d(w_{2r}) > d_P(w_{2r})$ (A similar argument holds for $d(w_1) > d_P(w_1)$.) Then, by the maximality of P, $d(w_1) = d_P(w_1)$ Since $d_P(w_1) = d(w_1) \geq 2$ (Lemma 1), there exists some $w_k \in P$ such that w_k is adjacent to w_1 and $k \neq 2$ Also, we can assume that k is the largest number among all such paths (with $d(w_{2r}) > d_P(w_{2r})$) We claim that either $k = 2r$ or the vertices of P form a cycle. Suppose this is not true If $k + 2 = 2r$, then $\text{dist}(w_1, w_{2r}) = 3$ and so $d(w_1) + d(w_{2r}) \geq n + 1$ Hence by Lemma 3, the vertices of P form a cycle, which is impossible. Thus $k + 2 \neq 2r$ and so $k \neq 2r - 2$. Since k is the largest number, it follows that $k < 2r - 2$ Hence $k + 2 < 2r$.

We claim that, if $w_i \in D_P(w_1)$ and $i \neq 2$, then $w_{i-1} \notin D_p(w_{k+2})$ Suppose this is not true Since k is the largest number, $4 \leq i \leq k$ and so $w_{i-1}, w_{i-2}, ..., w_2, w_1, w_i, w_{i+1}, ..., w_{2r}$ is a path of length $2r$ Since $d(w_{2r}) > d_P(w_{2r})$, by the maximality of P, we have $d_P(w_{i-1}) = d(w_{i-1})$ But w_{i-1} is adjacent to w_{k+2} Therefore k is not the largest number among all such paths, which is a contraction Hence, if $w_i \in D_P(w_1)$ and $i \neq 2$, then $w_{i-1} \notin D_P(w_{k+2})$. Since w_{k+2} is not adjacent to w_1, we have

$$d_P(w_{k+2}) \leq (r-1) - (d_P(w_1) - 1) = r - d_P(w_1).$$

Hence $d_P(w_1) + d_P(w_{k+2}) \leq r$. Since $\text{dist}(w_1, w_{k+2}) = 3$, we have $d(w_1) + d(w_{k+2}) \geq n + 1$ Since $d_P(w_1) = d(w_1)$, the proof of Lemma 3, we have $d_P(w_1) + d_P(w_{k+2}) \geq r + 1$, which is a contraction Therefore either $k = 2r$, in which case we have a cycle, or the vertices of P form a cycle of length $2r$

We now have the main result of this paper.

THEOREM 7. Let G be a connected bipartite graph with bipartition (X, Y) such that $|X| \geq |Y| (\geq 2)$. Let $n = |X|$ and $m = |Y|$ Suppose, for all vertices $x \in X$ and $y \in Y$, $\text{dist}(x, y) = 3$ implies $d(x) + d(y) \geq n + 1$. Then G contains a cycle of length $2m$ In particular, if $m = n$, then G is hamiltonian

PROOF. Let $P = (w_1, w_2, ..., w_{2r})$ be a longest even path in G By Lemma 6, we can assume that w_1 is adjacent to w_{2r} We show that $r = m$. Suppose that this is not true Then $r < m$ and so $n \geq m \geq r + 1$. Let $u \in X - P$ and $v \in Y - P$ Since G is connected, there exists a shortest path Q from u to P If $|Q| > 1$, then there exists an even path with length greater than $2r$ in G, which is impossible Hence $|Q| = 1$ and so $d_P(u) = d(u) \geq 2$. Similarly $d_P(v) = d(v) \geq 2$ In particular, u is not adjacent to v.

If there exists some $w_i \in P$ such that w_i is adjacent to u and w_{i+1} (or w_{i-1}) is adjacent to v, then we have an even path of length greater than $2r$, which is impossible. Therefore $d(u) + d(v) \leq r$ and $\text{dist}(u, v) > 3$ We can assume that $d(u) \geq d(v)$ (A similar argument holds for $d(v) \geq d(u)$) Since $d(u) \geq 2$ and $d(u) + d(v) \leq r$, there exists some vertex, say w_3, such that w_3 is adjacent to v and w_1 is not adjacent to v Since $\text{dist}(u, v) > 3$, w_2 is not adjacent to u Since $\text{dist}(w_1, v) = 3$, $d(w_1) + d(v) \geq n + 1$ Since $d(v) = d_P(v)$, by the proof of Lemma 3, $d_P(w_1) + d_P(v) \geq r + 1$ Hence

$$d_P(w_1) + d_P(u) \geq d_P(w_1) + d_P(v) \geq r + 1.$$

Thus there exists some vertex $w_i \in P$ such that w_i is adjacent to both u and w_1. It follows that $\text{dist}(w_2, u) = 3$ and so $d(w_2) + d(u) \geq n + 1$ If $d(w_2) > d_P(w_2)$, then we clearly have an even path of length greater than $2r$, because v is adjacent to w_3. But this is impossible. Hence $d(w_2) = d_P(w_2)$ and so $d_P(w_2) + d_P(u) = d(w_2) + d(u) \geq n + 1$. Since $n \geq r + 1$, we have $d_P(w_2) + d_P(u) \geq r + 2$ Therefore there exists some k and k' with $k' > k$ such that w_k and $w_{k'}$ are adjacent to u and either w_{k+1} is adjacent to w_2 or $w_{k'-1}$ is adjacent to w_2, otherwise $d_P(w_2) \leq r - (d_P(u) - 1)$ and so $d_P(w_2) + d_P(u) \leq r + 1$ If w_{k+1} is adjacent to w_2, then we have $w_{k'+1}, w_{k'+2}, ..., w_1, w_2, w_{k+1}, w_{k+2}, ..., w_{k'}, u, w_k, w_{k-1}, ..., w_3, v$ and this is a path of length $2r + 2$, which is impossible. If $w_{k'-1}$ is adjacent to w_2, then we have $w_{k+1}, w_{k+2}, ..., w_{k'-1}, w_2, w_1, ..., w_{k'}, u, w_k, w_{k-1}, ..., w_3, v$ and this is also a path of length $2r + 2$. This is a contraction Hence $r = m$ and this completes the proof of the theorem

We have the last result of this section.

COROLLARY 8. G is 2-connected

PROOF. By Theorem 7, G contains a cycle P of length $2m$ If $m = n$, then G is hamiltonian and so G is 2-connected Suppose $m < n$ For each vertex $x \in X - P$, we have $d_P(x) = d(x) \geq 2$ Hence, it follows that G is also 2-connected.

3. SOME REMARKS

In this section, we give some remarks

REMARK 1. The condition "dist$(x, y) = 3$ implies $d(x) + d(y) \geq n + 1$" in Theorem 7 cannot be replaced by a weaker condition "dist$(x, y) = 3$ implies $\max(d(x), d(y)) \geq (n + 1)/2$. In fact, let G be the graph given in Figure 1, where the vertex partition is indicated by the filled and empty circles

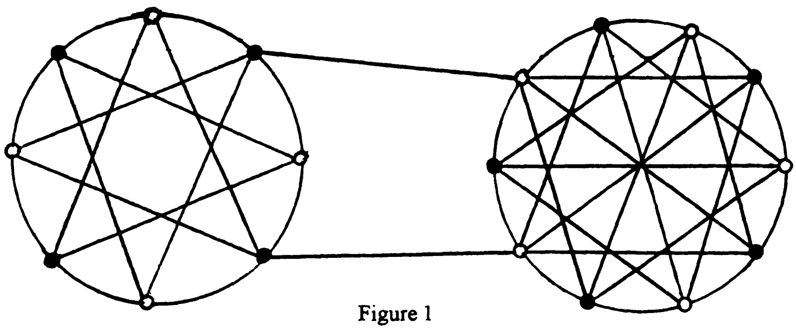

Figure 1

Then $n = m = 9$ Clearly G is not hamiltonian and G satisfies the condition dist$(x, y) = 3$ implies $\max(d(x), d(y)) \geq (n + 1)/2$

REMARK 2. The condition "dist$(x, y) = 3$ implies $d(x) + d(y) \geq n + 1$" in Theorem 7 cannot be replaced by the condition "dist$(x, y) = 3$ implies $d(x) + d(y) \geq m + 1$," if $m \neq n$ In fact, let G be the graph given in Figure 2

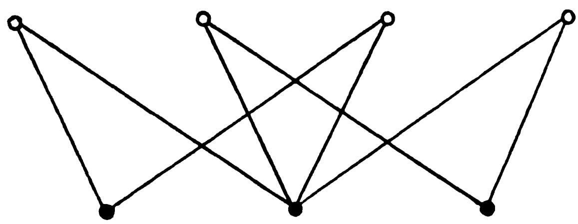

Figure 2

Then $n = 4$ and $m = 3$. Also $d(x) + d(y) \geq m + 1 = 4$ for all $x \in X$ and $y \in Y$ But G contains no cycle of length $2m = 6$. Since G has a cut vertex, G is not 2-connected.

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