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θ-ALMOST SUMMABLE SEQUENCES

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ABSTRACT King[3] introduced and examined the concepts of almost A-summable sequence, almost conservative matrix and almost regular matrix By following King, in this paper we introduce and examine the concepts of θ-almost A-summable sequence, θ-almost conservative matrix and θ-almost regular matrix

KEY WORDS AND PHRASES Lacunary sequence, θ-almost summable sequence, θ-almost conservative matrix, θ-almost regular matrix
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1. INTRODUCTION

Let A=(a_{nk}) be an infinite matrix of complex numbers and let x=$\{x_k\}$ be a sequence of complex numbers. The sequence $\{A_n(x)\}$ defined by

$$A_n(x)=\sum_{k=0}^{\infty}a_{nk}x_k$$

is called the A-transform of x whenever the series converges for n=1,2,3, The sequence x is said to be A-summable to L if $\{A_n(x)\}$ converges to L

Let **m** denote the linear space of bounded sequences A sequence $x\in$ **m** is said to be almost converget to L if

$$\lim_{m\to\infty}\sum_{k=1}^{m}x_{k+i}=L \text{ uniformly in } i.$$

A sequence $x\in$ **m** is said to be almost A-summable to L if the A-transform of x is almost converget to L. The matrix A is said to be almost conservative if $x\in$ **c** implies that the A-transform of x is almost convergent A is said to be almost regular if the A-transform of x almost convergent to the limit of x for each $x\in$ **c** , where **c** is the linear space of convergent sequences.

In the sequel the following notation is used· **C** denotes the complex numbers, **Z** denotes the integers and **N** denotes positive integers. If x=$\{x_k\}$ is an elemet of **c**, then $\|x\|$ is defined by $\|x\|=\sup\{|x_k| \quad k\in N\}$. The linear space of all continuous linear functionals on **c** is denoted by **c***. We use the special sequences e=(1,1, ..), e_k=(0, .,0,1,0,) (with 1 in rank k) and denote $\Delta=\{e,e_0,e_1,e_2, ..\}$.

By a lacunary sequence we shall mean an increasing integer sequence $\theta=(k_r)$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$ Throughout this paper the intervals determined by $\theta=(k_r)$ will be denoted by $I_r=(k_{r-1},k_r]$.

Freedman, Sember and Raphael[2] defined the space N_θ in the following manner: For any lacunary sequence $\theta=(k_r)$,

$$N_\theta=\left\{x=(x_k). \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|x_k-L|=0 \text{ , for some L}\right\}$$

Quite recently, the space M_θ was introduced by Das and Mishra[1] as below

$$M_\theta=\left\{x=(x_k): \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|x_{k+m}-L|=0 \text{ , for some L, uniformly in m}\right\}$$

King[3], introduced and examined the concepts of almost A-summable sequence, almost conservative matrix and almost regular matrix

In the present note, we introduce and examine the concepts of θ-almost A-summable sequence, θ-almost conservative matrix and θ-almost regular matrix.

2. DEFINITIONS AND THEOREMS

Definition 1 A sequence x is said to be θ-almost convergent to L if, for any lacunary sequence θ,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}x_{k+m}=L \text{ uniformly in m}$$

Definition 2. A sequence x is said to be θ-almost A-summable to L if, for any lacunary sequence θ, the A-transform of x is θ-almost convergent to L.

Definition3. The matrix A is said to be θ-almost conservative if $x\in \mathbf{c}$ implies that the A-transform of x is θ-almost convergent . A is said to be θ-almost regular if the A-transform of x is θ-almost convergent to the limit of **x** for each $x\in \mathbf{c}$.

Theorem 1. Let $A=(a_{nk})$ be an infinite matrix and let θ be a lacunary sequence.Then the matrix A is θ-almost conservative if and only if

(i) $$\sup_r\left\{\sum_{k=0}^{\infty}\frac{1}{h_r}\left|\sum_{k\in I_r}a_{n+j,k}\right| \ : \ r\in N\right\}\prec\infty\ ,\quad n=0,1,2,\ldots ,$$

(ii) there exists an $a\in \mathbf{C}$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{j\in I_r}\sum_{k=0}^{\alpha}a_{n+j,k}=a \text{ uniformly in n, and}$$

(iii) there exists an $a_k\in \mathbf{C}$, k=0,1,2,. . , such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{j\in I_r}a_{n+j,k}=a_k \text{ uniformly in n.}$$

Proof. Suppose that A is θ-almost conservative.Fix $n\in N$. Let

$$t_m(x)=\frac{1}{h_r}\sum_{j\in I_r}w_{j+n}(x),$$

where $w_{j+n}(x)=\sum_{k=0}^{\infty}a_{j+n,k}x_k$.

It is clear that $w_{j+n}\in c^*$, j=n=1,2,.... Hence $t_m\in c^*$ Since A is θ-almost conservative

$$\lim_{r\to\infty}t_m(x)=t(x)$$ uniformly in n. It follows that $\{t_m(x)\}$ is bounded for $x\in \mathbf{c}$ and fixed n

Hence $\{\|t_m\|\}$ is bounded by the uniform boundedness principle

For each $p\in N$, define the sequence v=v(n,r) by

$$v_k=\begin{cases}\operatorname{sgn}\sum_{j\in I_r}a_{j+n,k}, & 0\le k\le p\\ 0\ , & p\prec k\end{cases}$$

Then $v\in c$, $\|v\|=1$, and

$$|t_m(v)|=\frac{1}{h_r}\sum_{k=0}^{p}\left|\sum_{j\in I_r}a_{j+n,k}\right|$$

Hence $|t_m(v)|\le\|t_m\|\|v\|=\|t_m\|$ Therefore $\frac{1}{h_r}\sum_{k=0}^{\infty}\left|\sum_{j\in I_r}a_{n+j,k}\right|\le\|t_m\|$, so that (i) follows

Since e and e_k are convergent sequences, k=0,1,... , $\lim_r t_m(e)$ and $\lim_r t_m(e_k)$ must exist uniformly in n Hence (ii) and (iii) must hold

Now assume that

$$t_m(x)=\frac{1}{h_r}\sum_{j\in I_r}\sum_{k=0}^{\infty}a_{n+j,k}x_k=\frac{1}{h_r}\sum_{k=0}^{\infty}\sum_{j\in I_r}a_{n+j,k}x_k$$

so that

$$|t_m(x)|\le\frac{1}{h_r}\sum_{k=0}^{\infty}\left|\sum_{j\in I_r}a_{n+j,k}\right|\|x\|\quad j,n=1,2,\ldots$$

Therefore, $|t_{rn}(x)| \le K_n\|x\|$ by (i), where K_n is a coctant independent of r Hence $t_{rn} \in c^*$, and the sequence $\{\|t_{rn}\|\}$ is bounded for each n (ii) and (iii) imply that $\lim_r t_{rn}(e)$ and $\lim_r t_{rn}(e_k)$ exist for n=k=1,2, Since Δ is a fundamental set in **c** it follows from an elementary result of functional analysis that $\lim_r t_{rn}(x)=t_n(x)$ exists and $t_n \in$ **c***. Therefore t_n has the form

$$t_n(x)=b\left[t_n(e)-\sum_{k=0}^{\infty} t_n(e_k)\right]+\sum_{k=0}^{\infty} x_k t_n(e_k),$$

where $b=\lim_k x_k$ But $t_n(e)=a$ and $tn(e_k)=a_k$,k=0,1,2,.. , by(ii) and (iii), respsectively. Hence $\lim_r t_{rn}(x)=t(x)$ exists for each $x \in c$ n=0,1,2, . , with

$$t(x)=b\left[a-\sum_{k=0}^{\infty} a_k\right]+\sum_{k=0}^{\infty} a_k x_k \tag{1}$$

Since $t_{rn} \in c^*$ for each r and n, it has the form

$$t_{rn}(x)=b\left[t_{rn}(e)-\sum_{k=0}^{\infty} t_{rn}(e_k)\right]+\sum_{k=0}^{\infty} x_k t_{rn}(e_k), \tag{2}$$

It is easy to see from (1) and (2) that the convergence of $\{t_{rn}(x)\}$ to t(x) is uniform in n, since $t_{rn}(e)\to a$ and $t_{rn}(e_k)\to a_k$ uniformly in n Therefore A is θ-almost conservative and the theorem is proved

Theorem 2 Let $A=(a_{nk})$ be an infinite matrix and let θ be a lacunary sequence Then the matrix A is θ-almost regular if and only if

(iv) $$\sup_r\left\{\sum_{k=0}^{\infty}\frac{1}{h_r}\left|\sum_{j\in I_r} a_{n+j,k}\right| \quad r\in N\right\}\prec\infty\ ,\quad n=0,1,2,\ ,$$

(v) $$\lim_{r\to\infty}\frac{1}{h_r}\sum_{j\in I_r,}\sum_{k=0}^{\infty} a_{n+j,k}=1 \quad \text{uniformly in n, and}$$

(vi) $$\lim_{r\to\infty}\frac{1}{h_r}\sum_{j\in I_r} a_{n+j,k}=0 \quad \text{uniformly in n , k = 0,1,2,}$$

Proof. Suppose that A is θ-almost regularThen A is θ-almost conservative so that (iv) must hold by Theorem 1 (v) and (vi) must hold since the A-transform of the sequences e_k and e must be θ-almost convergent to 0 and 1, respectively.

Now suppose that (iv),(v) and (vi) hold Then A is θ-almost concervative by Theorem 1 Therefore $\lim_r t_{rn}(x)=t(x)$ uniformly in n for each $x\in c$ The representation (1) gives $t(x)=\lim_k x_k$. Hence A is θ-almost regular. This proves the theorem

In the applications of summability theory to function theory it is important to know the region in which the sequence of partial sums of the geometric series is A-summable to 1/(1-z) for a given matrix A The following theorem is helpful in determining the region in which the sequence of partial sums of the geometric series is θ-almost A-summable to 1/(1-z)

Theorem 3. Let A be an infinite matrix such that (v) holds The sequence of partial sums of the geometric series is θ-almost A-summable to 1/(1-z) if and only if $z\in\Omega$ where

$$\Omega=\left\{z:\ \lim_{r\to\infty}\frac{1}{h_r}\sum_{j\in I_r,}\sum_{k=0}^{\infty} a_{n+j,k}z^k=0 \quad \text{uniformly in n}\right\}$$

Proof. Let $\{s_k(z)\}$ denote the sequence of partial sums of the geometric series Then

$$t_{rn}(x)=\frac{1}{h_r}\sum_{j\in I_r,}\sum_{k=0}^{\infty} a_{n+j,k}s_k(z)=\ \frac{1}{h_r}\sum_{j\in I_r,}\sum_{k=0}^{\infty} a_{n+j,k}\left[\frac{1-z^{k+1}}{1-z}\right]$$

Hence,

$$\lim_r t_{rn}=\frac{1}{1-z}-\lim_r\frac{z}{h_r(1-z)}\sum_{j\in I_r,}\sum_{k=0}^{\infty} a_{j+n,k}z^k$$

Therefore

$$\lim_r t_{rn}=\frac{1}{1-z}\quad \text{unifomly in n if and only if } z\in\Omega$$

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REFERENCES

[1] DAS , G and MISHRA, S.K.,"Banach limits and lacunary strong almost convergence," The J of the Orissa Math Soc Vol.2(1983),61-70

[2] FREEDMAN, A R , SEMBER, J J and RAPHAEL , M ,"Some Cesaro type summability spaces", Proc London Math Soc (3)37(1978),508-520

[3] KING, J P , " Almost summable sequences," Proc. Amer Math Soc 17(1966),1219-1225

[4] Lorentz, G G ,"A contribution to the theory of divergent sequences," Acta Math 80, (1948), 167-190.