C-COMPACTNESS MODULO AN IDEAL

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We investigate the concepts of quasi-$H$-closed modulo an ideal which generalizes quasi-$H$-closedness and $C$-compactness modulo an ideal which simultaneously generalizes $C$-compactness and compactness modulo an ideal. We obtain a characterization of maximal $C$-compactness modulo an ideal. Preservation of $C$-compactness modulo an ideal by functions is also investigated.

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1. Introduction

In the present paper, we consider a topological space equipped with an ideal, a theme that has been treated by Vaidyanathaswamy [15] and Kuratowski [6] in their classical texts. An ideal $\mathcal{I}$ on a set $X$ is a nonempty subset of $P(X)$, the power set of $X$, which is closed for subsets and finite unions. An ideal is also called a dual filter. $\{\phi\}$ and $P(X)$ are trivial examples of ideals. Some useful ideals are (i) $\mathcal{I}_f$, the ideal of all finite subsets of $X$, (ii) $\mathcal{I}_c$, the ideal of all countable subsets of $X$, (iii) $\mathcal{I}_n$, the ideal of all nowhere dense subsets in a topological space $(X, \tau)$, and (iv) $\mathcal{I}_s$, the set of all scattered sets in $(X, \tau)$. For an ideal $\mathcal{I}$ on $X$ and $A \subseteq X$, we denote the ideal $\{I \cap A : I \in \mathcal{I}\}$ by $\mathcal{I}_A$.

A topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$ is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subseteq X$, $A^*(\mathcal{I}, \tau)$ (called the adherence of $A$ modulo an ideal $\mathcal{I}$) or $A^*(\mathcal{I})$ or just $A^*$ is the set $\{x \in X : A \cap U \notin \mathcal{I} \text{ for every open neighborhood } U \text{ of } x\}$. $A^*(\mathcal{I}, \tau)$ has been called the local function of $A$ with respect to $\mathcal{I}$ in [6]. It is easy to see that (i) for the ideal $\{\phi\}$, $A^*$ is the closure of $A$, (ii) for the ideal $P(X)$, $A^*$ is $\phi$, and (iii) for ideal $\mathcal{I}_f$, $A^*$ is the set of all $\omega$-accumulation points of $A$. For general properties of the operator $*$, we refer the readers to [5, 14].

Observe that the operator $cl^*: P(X) \to P(X)$ defined by $cl^*(A) = A \cup A^*$ is a Kuratowski closure operator on $X$ and hence generates a topology $\tau^*(\mathcal{I})$ or just $\tau^*$ on $X$ finer than $\tau$. As has already been observed, $\tau^*(\{\phi\}) = \tau$ and $\tau^*(P(X)) = \text{the discrete topology}$. A description of open sets in $\tau^*(\mathcal{I})$ as given in Vaidyanathaswamy [15] is given in the following.
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**Theorem 1.1.** If $\tau$ is a topology and $\mathcal{I}$ is an ideal, both defined on $X$, then

$$\beta = \beta(\tau, \mathcal{I}) = \{ V - I : V \in \tau, I \in \mathcal{I} \}$$

is a base for the topology $\tau^*(\mathcal{I})$ on $X$.  

Ideals have been used frequently in the fields closely related to topology, such as real analysis, measure theory, and lattice theory. Some interesting illustrations of $\tau^*(\mathcal{I})$ are as follows [5].

1. If $\tau$ is the topology generated by the partition $\{ 2n - 1, 2n \} : n \in \mathbb{N} \}$ on the set $\mathbb{N}$ of natural numbers, then $\tau^*(\mathcal{I})$ is the discrete topology.
2. If $\tau$ is the indiscrete topology on a set $X$, then $\tau^*(\mathcal{I})$ is the cofinite topology on $X$, and $\tau^*(\mathcal{I}_c)$ is the co-countable topology on $X$. If for a fixed point $p \in X$, $\mathcal{I}$ denotes the ideal $\{ A \subset X : p \notin A \}$, then $\tau^*(\mathcal{I})$ is the particular point topology on $X$.
3. For any topological space $(X, \tau)$, $\tau^*(\mathcal{I}_n)$ is the $\tau^*$ topology of Njástad [10].
4. If $\tau$ is the usual topology on the real line $\mathbb{R}$ and $\mathcal{I}$ is the ideal of all subsets of $\mathbb{R}$ of Lebesgue measure zero, then $\tau^*$-Borel sets are precisely the Lebesgue measurable sets of $\mathbb{R}$.

2. **Quasi-$H$-closed modulo an ideal space**

The concept of compactness modulo an ideal was introduced by Newcomb [9] and has been studied among others by Rancin [11], and Hamlett and Janković [3]. A space $(X, \tau)$ is defined to be compact modulo an ideal $\mathcal{I}$ on $X$ or just $(\mathcal{I})$ compact space if for every open cover $\mathcal{U}$ of $X$, there is a finite subfamily $\{ U_1, U_2, \ldots, U_n \}$ such that $X - \bigcup_{i=1}^n U_i \in \mathcal{I}$. In this section, we define quasi-$H$-closedness modulo an ideal and study some of its properties. In the process, we get some interesting characterizations of quasi-$H$-closed spaces.

**Definition 2.1.** Let $(X, \tau)$ be a topological space and $\mathcal{I}$ an ideal on $X$. $X$ is quasi-$H$-closed modulo $\mathcal{I}$ or just $(\mathcal{I})$ QHC if for every open cover $\mathcal{U}$ of $X$, there is a finite subfamily $\{ U_1, U_2, \ldots, U_n \}$ of $\mathcal{U}$ such that $X - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. Such a subfamily is said to be proximate subcover modulo $\mathcal{I}$ or just $(\mathcal{I})$ proximate subcover.

A subset $A$ of a topological space $(X, \tau)$ is said to be preopen [8] if $A \subset \text{int}(\text{cl}(A))$. The collection of all preopen sets of a space $(X, \tau)$ is denoted by $\text{PO}(X)$. An ideal $\mathcal{I}$ of subsets of a topological space $(X, \tau)$ is said to be codense [1] if the complement of each of its members is dense. Note that an ideal $\mathcal{I}$ is codense if and only if $\mathcal{I} \cap \tau = \{ \phi \}$. Codense ideals are called $\tau$-boundary ideals in [9]. An ideal $\mathcal{I}$ of subsets of a topological space $(X, \tau)$ is said to be completely codense [1] if $\mathcal{I} \cap \text{PO}(X) = \{ \phi \}$. Obviously, every completely codense ideal is codense. Note that if $(\mathbb{R}, \tau)$ is the set $\mathbb{R}$ of real numbers equipped with the usual topology $\tau$, then $\mathcal{I}_c$ is codense but not completely codense ideal. It is proved in [1] that an ideal $\mathcal{I}$ is completely codense if and only if $\mathcal{I} \subset \mathcal{I}_c$.

From the discussion of Section 1, the proof of the following theorem is immediate.

**Theorem 2.2.** For a space $(X, \tau)$, the following are equivalent:

(a) $(X, \tau)$ is quasi-$H$-closed;
(b) \((X, \tau)\) is \((\{\phi\})\) QHC;
(c) \((X, \tau)\) is \((\mathcal{F})\) QHC;
(d) \((X, \tau)\) is \((\mathcal{F}_n)\) QHC;
(e) \((X, \tau)\) is \((\mathcal{F})\) QHC for every codense ideal \(\mathcal{F}\).

The significance of condition in (e) may be seen by considering the set \(\mathbb{R}\) of real numbers equipped with the usual topology \(\tau\). If \(A\) is a finite subset of \(\mathbb{R}\) and \(\mathcal{I}\) is the ideal of all subsets of \(\mathbb{R} - A\), then \((\mathbb{R}, \tau)\) is \((\mathcal{I})\) QHC, but not quasi-\(H\)-closed.

A family \(\mathcal{F}\) of subsets of \(X\) is said to have the finite-intersection property modulo an ideal \(\mathcal{I}\) on \(X\) or just \((\mathcal{I})\) FIP if the intersection of no finite subfamily of \(\mathcal{F}\) is a member of \(\mathcal{I}\).

Recall that a subset in a space is called regular open if it is the interior of its own closure. The complement of a regular open set is called regular closed. It is proved in [12] that for completely codense ideal \(\mathcal{F}\) on a space \((X, \tau)\), the collections of regular open sets of \((X, \tau)\) and \((X, \tau^*)\) are same. The following theorem contains a number of characterizations of \((\mathcal{F})\) QHC spaces. Since the proof is similar to that of a theorem in the next section, we omit it.

**Theorem 2.3.** For a space \((X, \tau)\) and an ideal \(\mathcal{F}\) on \(X\), the following are equivalent:

(a) \((X, \tau)\) is \((\mathcal{F})\) QHC;
(b) for each family \(\mathcal{F}\) of closed sets having empty intersection, there is a finite subfamily \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^n \text{int}(F_i) \subseteq \mathcal{I}\);
(c) for each family \(\mathcal{F}\) of closed sets such that \(\{\text{int}(F) : F \in \mathcal{F}\}\) has \((\mathcal{I})\) FIP, one has \(\bigcap\{F : F \in \mathcal{F}\} \neq \phi\);
(d) every regular open cover has a finite \((\mathcal{F})\) proximate subcover;
(e) for each family \(\mathcal{F}\) of nonempty regular closed sets having empty intersection, there is a finite subfamily \(\{F_1, F_2, F_3, \ldots, F_n\}\) such that \(\bigcap_{i=1}^n \text{int}(F_i) \subseteq \mathcal{I}\);
(f) for each collection \(\mathcal{F}\) of nonempty regular closed sets such that \(\{\text{int}(F) : F \in \mathcal{F}\}\) has \((\mathcal{I})\) FIP, one has \(\bigcap\{F : F \in \mathcal{F}\} \neq \phi\);
(g) for each open filter base \(\mathcal{B}\) on \(P(X) - \mathcal{I}\), \(\bigcap\{\text{cl}(B) : B \in \mathcal{B}\} \neq \phi\);
(h) every open ultrafilter on \(P(X) - \mathcal{I}\) converges.

It follows from a result in [13] that \(\tau\) and \(\tau^*(\mathcal{F})\) have the same regular open sets, where \(\mathcal{F}\) is a completely codense ideal on \((X, \tau)\). In particular, if \(U \in \tau^*\), then \(\text{cl}(U) = \text{cl}^*(U)\). Using this observation along with the previous theorem, we have the following.

**Theorem 2.4.** Let \(\mathcal{I}\) be a completely codense ideal on a space \((X, \tau)\). Then \((X, \tau)\) is \((\mathcal{I})\) QHC if and only if \((X, \tau^*)\) is \((\mathcal{I})\) QHC.

Combining this result with Theorem 2.2, we have the following.

**Corollary 2.5.** Let \((X, \tau)\) be a space and \(\mathcal{F}\) a completely codense ideal on \(X\). Then the following are equivalent:

(a) \((X, \tau)\) is quasi-\(H\)-closed;
(b) \((X, \tau^*)\) is quasi-\(H\)-closed;
(c) \((X, \tau^a)\) is quasi-\(H\)-closed.

The last equivalence follows because \(\tau^a = \tau^*(\mathcal{F}_n)\), where \(\mathcal{F}_n\) is the ideal of nowhere dense sets in \(X\).
In this section, we generalize the concepts of C-compactness of Viglino [16] and compactness modulo an ideal due to Newcomb [9] and Rancin [11]. A space $(X, \tau)$ is said to be C-compact if for each closed set $A$ and each $\tau$-open covering $\mathcal{U}$ of $A$, there exists a finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $A \subseteq \bigcup_{i=1}^{n} \text{cl}(U_i)$.

**Definition 3.1.** Let $(X, \tau)$ be a topological space and $\mathcal{I}$ an ideal on $X$. $(X, \tau)$ is said to be C-compact modulo $\mathcal{I}$ or just C($\mathcal{I}$)-compact if for every closed set $A$ and every $\tau$-open cover $\mathcal{U}$ of $A$, there is a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ such that $A - \bigcup_{i=1}^{n} \text{cl}(U_i) \in \mathcal{I}$.

It follows from the definition that

- compact (\mathcal{I})-compact
- C-compact C(\mathcal{I})-compact
- quasi-H-closed (\mathcal{I})\text{QHC}

(3.1)

Also from the definition in Section 1, we have the following.

**Theorem 3.2.** For a space $(X, \tau)$, the following are equivalent:

(a) $(X, \tau)$ is C-compact;
(b) $(X, \tau)$ is C($\phi$)-compact;
(c) $(X, \tau)$ is C($\mathcal{I}_f$)-compact.

**Example 3.3.** For $n$ and $m$ in the set $N$ of positive integers, let $Y$ denote the subset of the plane consisting of all points of the form $(1/n, 1/m)$ and the points of the form $(1/n, 0)$. Let $X = Y \cup \{\infty\}$. Topologize $X$ as follows: let each point of the form $(1/n, 1/m)$ be open. Partition $N$ into infinitely many infinite-equivalence classes, $\{Z_i\}_{i=1}^{\infty}$. Let a neighborhood system for the point $(1/i, 0)$ be composed of all sets of the form $G \cup F$, where

\[ G = \left\{ \left( \frac{1}{i}, 0 \right) \right\} \cup \left\{ \left( \frac{1}{i}, \frac{1}{m} \right) : m \geq k \right\}, \]
\[ F = \left\{ \left( \frac{1}{n}, \frac{1}{m} \right) : m \in Z_i, n \geq k \right\} \]

for some $k \in N$. Let a neighborhood system for the point $\infty$ be composed of sets of the form $X \setminus T$, where

\[ T = \left\{ \left( \frac{1}{n}, 0 \right) : n \in N \right\} \cup \bigcup_{i=1}^{k} \left\{ \left( \frac{1}{i}, \frac{1}{m} \right) : m \in N \right\} \cup \left\{ \left( \frac{1}{n}, \frac{1}{m} \right) : m \in Z_i, n \in N \right\} \]

for some $k \in N$. It is shown in [16] that $X$ is a C-compact space which is not compact. In view of Theorem 3.2, such a space is C($\mathcal{I}_f$)-compact, but not ($\mathcal{I}_f$) compact.
Example 3.4. Let \( X = \mathbb{R}^+ \cup \{a\} \cup \{b\} \), where \( \mathbb{R}^+ \) denotes the set of nonnegative real numbers and \( a, b \) are two distinct points not in \( \mathbb{R}^+ \). Let \( W(a) = \{ V \subset X : V = \{a\} \cup \bigcup_{r=m}^\infty (2r, 2r+1) \} \), where \( m \) is a nonnegative integer, be a neighborhood system for the point \( a \). Let \( W(b) = \{ V \subset X : V = \{b\} \cup \bigcup_{r=m}^\infty (2r-1, 2r) \} \), where \( m \) is a nonnegative integer, be a neighborhood system for the point \( b \). Let \( R^+ \), with the usual topology, be imbedded in \( X \). Viglino [16] has shown that the space \( X \) is not \( C \)-compact. If \( A \) is a finite subset of \( X \), then \((X, \tau)\) is \( C(\mathcal{J}) \)-compact, where \( \mathcal{J} \) is the ideal of all subsets of \( X - A \).

In view of Examples 3.3 and 3.4, it is clear that the implications shown after Definition 3.1 are, in general, irreversible.

It is proved in [3] that if \((X, \tau)\) is quasi-\( H \)-closed and \( \mathcal{J} \) is an ideal such that \( \mathcal{J}_n \subset \mathcal{J} \), then \((X, \tau)\) is \((\mathcal{J})\) compact (and hence \( C(\mathcal{J}) \)-compact).

Next, if \( \{U_1, U_2, \ldots, U_n\} \) is a finite collection of open subsets such that \( X - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{J}_n \), then \( X - \bigcup_{i=1}^n \text{cl}(U_i) = \emptyset \) because \( \tau \cap \mathcal{J}_n = \{\emptyset\} \). But then \( \text{int}(\text{cl}(X - \bigcup_{i=1}^n U_i)) = X - \bigcup_{i=1}^n \text{cl}(U_i) = \emptyset \) implies that \( X - \bigcup_{i=1}^n U_i \in \mathcal{J}_n \). Therefore, a space \((X, \tau)\) is \((\mathcal{J}_n)\) compact if and only if it is \( C(\mathcal{J}_n) \)-compact. In view of this discussion, we have the following.

**Theorem 3.5.** For a space \((X, \tau)\), the following are equivalent:

a) \((X, \tau)\) is quasi-\( H \)-closed;

b) \((X, \tau)\) is \((\mathcal{J}_n)\) QHC;

c) \((X, \tau)\) is \( C(\mathcal{J}_n) \)-compact;

d) \((X, \tau)\) is \((\mathcal{J}_n)\) compact.

A space \((X, \tau)\) is said to be Baire if the intersection of every countable family of open sets in \((X, \tau)\) is dense. It is noted in [5] that a space \((X, \tau)\) is Baire if and only if \( \tau \cap \mathcal{J}_m = \{\emptyset\} \), where \( \mathcal{J}_m \) is the ideal of meager (first category) subsets of \((X, \tau)\). Thus, in view of the above theorem, a Baire space \((X, \tau)\) is \( C(\mathcal{J}_m) \)-compact if and only if it is quasi-\( H \)-closed.

We now give some characterizations of \( C(\mathcal{J}) \)-compact spaces.

**Theorem 3.6.** Let \((X, \tau)\) be a space and let \( \mathcal{J} \) be an ideal on \( X \). Then the following are equivalent:

a) \((X, \tau)\) is \( C(\mathcal{J}) \)-compact;

b) for each closed subset \( A \) of \( X \) and each family \( \mathcal{F} \) of closed subsets of \( X \) such that \( \bigcap \{F \cap A : F \in \mathcal{F}\} = \emptyset \), there exists a finite subfamily \( \{F_1, F_2, F_3, \ldots, F_n\} \) such that \( \bigcap \text{int}(F_i) \cap A \in \mathcal{J} \);

c) for each closed set \( A \) and each family \( \mathcal{F} \) of closed sets such that \( \{\text{int}(F) \cap A : F \in \mathcal{F}\} \) has \( (\mathcal{J}) \) FIP, one has \( \bigcap \{F \cap A : F \in \mathcal{F}\} \neq \emptyset \);

d) for each closed set \( A \) and each regular open cover \( \mathcal{U} \) of \( A \), there exists a finite subcollection \( \{U_1, U_2, U_3, \ldots, U_n\} \) such that \( A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{J} \);

e) for each closed set \( A \) and each family \( \mathcal{F} \) of regular closed sets such that \( \bigcap_1 F \cap A : F \in \mathcal{F}\} = \emptyset \), there is a finite subfamily \( \{F_1, F_2, F_3, \ldots, F_n\} \) such that \( \bigcap_{i=1}^n \text{int}(F_i) \cap A \in \mathcal{J} \);

f) for each closed set \( A \) and each family \( \mathcal{F} \) of regular closed sets such that \( \{\text{int}(F) \cap A : F \in \mathcal{F}\} \) has \( (\mathcal{J}) \) FIP, one has \( \bigcap \{F \cap A : F \in \mathcal{F}\} \neq \emptyset \);

g) for each closed set \( A \), each open cover \( \mathcal{U} \) of \( X - A \) and each open neighborhood \( V \) of \( A \), there exists a finite subfamily \( \{U_1, U_2, U_3, \ldots, U_n\} \) of \( \mathcal{U} \) such that \( X - (V \cup (\bigcup_{i=1}^n \text{cl}(U_i))) \in \mathcal{J} \);
(h) for each closed set $A$ and each open filter base $\mathcal{B}$ on $X$ such that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$, one has $\bigcap \{\text{cl}(B) : B \in \mathcal{B}\} \cap A \neq \emptyset$.

Proof. (a) $\Rightarrow$ (b). Let $(X, \tau)$ be $C(\mathcal{I})$-compact, $A$ a closed subset, and $\mathcal{F}$ a family of closed subsets with $\cap \{F \cap A : F \in \mathcal{F}\} = \emptyset$. Then $\{X - F : F \in \mathcal{F}\}$ is an open cover of $A$ and hence admits a finite subfamily $\{X - F_i : i = 1, 2, \ldots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - F_i) \in \mathcal{F}$. This set in $\mathcal{I}$ is easily seen to be $\bigcap_{i=1}^n \{\text{int}(F_i) \cap A\}$.

(b) $\Rightarrow$ (c). This is easy to be established.

(c) $\Rightarrow$ (a). Let $A$ be a closed subset, let $\mathcal{U}$ be an open cover of $A$ with the property that for no finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$, one has $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. Then $\{X - U : U \in \mathcal{U}\}$ is a family of closed sets. Since
\[
\bigcap_{i=1}^n \{X - \text{cl}(U_i)\} \cap A = \bigcap_{i=1}^n \{A - \text{cl}(U_i)\} = A - \bigcup_{i=1}^n \text{cl}(U_i),
\]
the family $\{\text{int}(X - U) \cap A : U \in \mathcal{U}\}$ has $\mathcal{I}$ FIP. By the hypothesis $\bigcap \{(X - U) \cap A : U \in \mathcal{U}\} \neq \emptyset$. But then $A - \bigcup \{U : U \in \mathcal{U}\} \neq \emptyset$, that is, $\mathcal{U}$ is not a cover of $A$, a contradiction.

(d) $\Rightarrow$ (a). Let $A$ be a closed subset of $X$ and $\mathcal{U}$ an open cover of $A$. Then $\{\text{int}(\text{cl}(U)) : U \in \mathcal{U}\}$ is a regular open cover of $A$. Let $\{\text{int}(\text{cl}(U_i)) : i = 1, 2, \ldots, n\}$ be a finite subfamily such that $A - \bigcup_{i=1}^n \text{cl}(\text{int}(\text{cl}(U_i))) \in \mathcal{I}$. Since $U_i$ is open and for each open set $U$, $\text{cl}(\text{int}(\text{cl}(U))) = \text{cl}(U)$, we have $A - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$, which shows that $X$ is $C(\mathcal{I})$-compact.

(a) $\Rightarrow$ (d). This is obvious.

The proofs for (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (d) are parallel to (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a), respectively.

(a) $\Rightarrow$ (g). Let $A$ be a closed set, $V$ an open neighborhood of $A$, and $\mathcal{U}$ an open cover of $X - A$. Since $X - V \subset X - A$, $\mathcal{U}$ is also an open cover of the closed set $X - V$.

Let $\{U_1, U_2, U_3, \ldots, U_n\}$ be a finite subcollection of $\mathcal{U}$ such that $(X - V) - \bigcup_{i=1}^n \text{cl}(U_i) \in \mathcal{I}$. However, the last set is $X - (V \cup \{\bigcup_{i=1}^n \text{cl}(U_i)\})$.

(g) $\Rightarrow$ (a). Let $A$ be a closed subset of $X$ and $\mathcal{U}$ an open covering of $A$. If $H$ denotes the union of members of $\mathcal{U}$, then $F = X - H$ is a closed set and $X - A$ is an open neighborhood of $F$. Also $\mathcal{U}$ is an open cover of $X - F$. By hypothesis, there is a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that
\[
X \left((X - A) \cup \left\{\bigcup_{i=1}^n \text{cl}(U_i)\right\}\right) \in \mathcal{I}.
\]
However, this set in $\mathcal{I}$ is nothing but $A - \bigcup_{i=1}^n \text{cl}(U_i)$.

(a) $\Rightarrow$ (h). Suppose $A$ is a closed set and $\mathcal{B}$ is any open filter base on $X$ with $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$. Suppose, if possible, $\bigcap \{\text{cl}(B) : B \in \mathcal{B}\} \cap A = \emptyset$. Then $\{X - \text{cl}(B) : B \in \mathcal{B}\}$ is an open cover of $A$. By the hypothesis, there exists a finite subfamily $\{X - \text{cl}(B_i) : i = 1, 2, 3, \ldots, n\}$ such that $A - \bigcup_{i=1}^n \text{cl}(X - \text{cl}(B_i)) \in \mathcal{I}$. However, this set is $A \cap (\bigcap_{i=1}^n \text{int(\text{cl}(B_i)))}$ and $A \cap (\bigcap_{i=1}^n B_i)$ is a subset of it. Therefore, $A \cap (\bigcap_{i=1}^n B_i) \in \mathcal{I}$. Since $\mathcal{B}$ is a filter base, we have a $B \in \mathcal{B}$ such that $B \subset \bigcap_{i=1}^n B_i$. But then $A \cap B \in \mathcal{I}$ which contradicts the fact that $\{B \cap A : B \in \mathcal{B}\} \subset P(X) - \mathcal{I}$.

(h) $\Rightarrow$ (a). Suppose that $(X, \tau)$ is not $C(\mathcal{I})$-compact. Then there exist a closed subset $A$ of $X$ and an open cover $\mathcal{U}$ of $A$ such that for any finite subfamily $\{U_1, U_2, U_3, \ldots, U_n\}$
of \( \mathcal{U} \), we have \( A - \bigcup_{i=1}^{n} \text{cl}(U_i) \notin \mathcal{F} \). We may assume that \( \mathcal{U} \) is closed under finite unions. Then the family \( \mathcal{B} = \{X - \text{cl}(U) : U \in \mathcal{U}\} \) is an open filter base on \( X \) such that \( \{B \cap A : B \in \mathcal{B}\} \subset P(A) - \mathcal{F} \). So, by the hypothesis, \( \bigcap \{\text{cl}(X - \text{cl}(U)) : U \in \mathcal{U}\} \cap A \neq \phi \). Let \( x \) be a point in the intersection. Then \( x \in A \) and \( x \in \text{cl}(X - \text{cl}(U)) = X - \text{int}(\text{cl}(U)) \subset X - U \) for each \( U \in \mathcal{U} \). But this contradicts the fact that \( \mathcal{U} \) is a cover of \( A \). Hence \((X, \tau)\) is \( C(\mathcal{F})\)-compact.

Next we characterize \( C(\mathcal{F})\)-compact spaces using some weaker forms of filter base convergence.

**Definition 3.7.** A filter base \( \mathcal{B} \) is said to be \( (\mathcal{F}) \) adherent convergent if for every neighborhood \( G \) of the adherent set of \( \mathcal{B} \), there exists an element \( B \in \mathcal{B} \) such that \((X - G) \cap B \in \mathcal{J}\). Clearly, every adherent convergent filter base is \( (\mathcal{F}) \) adherent convergent and a filter base is adherent convergent if and only if it is \((\{\phi\})\) adherent convergent.

**Theorem 3.8.** A space \((X, \tau)\) is \( C(\mathcal{F})\)-compact if and only if every open filter base on \( P(X) - \mathcal{F} \) is \( (\mathcal{F}) \) adherent convergent.

**Proof.** Let \((X, \tau)\) be \( C(\mathcal{F})\)-compact and let \( \mathcal{B} \) be an open filter base on \( P(X) - \mathcal{F} \) with \( A \) as its adherent set. Let \( G \) be an open neighborhood of \( A \). Then \( A = \bigcap \{\text{cl}(B) : B \in \mathcal{B}\}, A \subset G \), and \( X - G \) is closed. Now \( \{X - \text{cl}(B) : B \in \mathcal{B}\} \) is an open cover of \( X - G \) and so by the hypothesis, it admits a finite subfamily \( \{X - \text{cl}(B_i) : i = 1, 2, 3, \ldots, n\} \) such that \((X - G) - \bigcup_{i=1}^{n} \text{cl}(X - \text{cl}(B_i)) \in \mathcal{F} \). But this implies \((X - G) \cap \bigcap_{i=1}^{n} \text{int}(\text{cl}(B_i)) \in \mathcal{F} \). However, \( B_i \subset \text{int}(\text{cl}(B_i)) \) implies \((X - G) \cap \bigcap_{i=1}^{n} B_i \in \mathcal{F} \). Since \( \mathcal{B} \) is a filter base and \( B_i \in \mathcal{B} \), there is a \( B \in \mathcal{B} \) such that \( B \subset \bigcap_{i=1}^{n} B_i \). But then \((X - G) \cap B \in \mathcal{F} \) is required.

Conversely, let \((X, \tau)\) be not \( C(\mathcal{F})\)-compact, and let \( A \) be a closed set, and \( \mathcal{U} \) an open cover of \( A \) such that for no finite subfamily \( \{U_1, U_2, U_3, \ldots, U_n\} \) of \( \mathcal{U} \), one has \( A - \bigcup_{i=1}^{n} \text{cl}(U_i) \notin \mathcal{F} \). Without loss of generality, we may assume that \( \mathcal{U} \) is closed for finite unions. Therefore, \( \mathcal{B} = \{X - \text{cl}(U) : U \in \mathcal{U}\} \) becomes an open filter base on \( P(X) - \mathcal{F} \). If \( x \) is an adherent point of \( \mathcal{B} \), that is, if \( x \in \bigcap \{\text{cl}(X - \text{cl}(U)) : U \in \mathcal{U}\} = X - \bigcup \{\text{int}(\text{cl}(U)) : U \in \mathcal{U}\} \), then \( x \notin A \), because \( \mathcal{U} \) is an open cover of \( A \) and for \( U \in \mathcal{U} \), \( U \subset \text{int}(\text{cl}(U)) \). Therefore, the adherent set of \( \mathcal{B} \) is contained in \( X - A \), which is an open set. By the hypothesis, there exists an element \( B \in \mathcal{B} \) such that \( (X - (X - A)) \cap B \in \mathcal{I} \), that is, \( A \cap B \notin \mathcal{F} \), that is, \( A \cap (X - \text{cl}(U)) \notin \mathcal{F} \), that is, \( A - \text{cl}(U) \notin \mathcal{F} \) for some \( U \in \mathcal{U} \). This however contradicts our assumption. This completes the proof.

Herrington and Long [4] characterized \( C(\mathcal{F})\)-compact spaces using \( r\)-convergence of filters and nets. We obtain similar results for \( C(\mathcal{F})\)-compact spaces in the next definition.

**Definition 3.9.** Let \( X \) be a space, \( \phi \neq A \subset X \), and let \( \mathcal{B} \) be a filter base on \( A \). \( \mathcal{B} \) is said to \( r\)-converge to \( a \in A \) if for each open set \( V \) in \( X \) containing \( a \), there is \( B \in \mathcal{B} \) with \( B \subset \text{cl}(V) \). The filter base \( \mathcal{B} \) is said to \( r\)-accumulate to \( a \), if for each open set \( V \) containing \( a \), \( \text{cl}(V) \cap B \neq \phi \) for each \( B \in \mathcal{B} \).

Similarly, a net \( \varphi : D \to A \subset X \) is said to \( r\)-converge to \( a \in A \) if for each open set \( V \) containing \( a \), there is \( b \in D \) such that \( \varphi(c) \in \text{cl}(V) \) for all \( c \geq b \). \( \varphi \) is said to \( r\)-accumulate to \( a \) if for each open set \( V \) containing \( a \) and each \( b \in D \), there is \( c \in D \) with \( c \geq b \) and \( \varphi(c) \in \text{cl}(V) \).
It is known [4] that convergence (accumulation) for filter bases and nets implies r-convergence (r-accumulation), but the converse is not true.

**Theorem 3.10.** For a space \((X, \tau)\) and an ideal \(\mathfrak{I}\) on \(X\), the following are equivalent:

(a) \((X, \tau)\) is \(C(\mathfrak{I})\)-compact;

(b) for each closed set \(A\), each filter base \(\mathcal{B}\) on \(P(A) - \mathfrak{I}\) r-accumulates to some \(a \in A\);

(c) for each closed set \(A\), each maximal filter base \(\mathcal{M}\) on \(P(A) - \mathfrak{I}\) r-converges to some \(a \in A\);

(d) for each closed set \(A\), each net \(\varphi\) on \(P(A) - \mathfrak{I}\) r-accumulates to some \(a \in A\).

**Proof.** (a) \(\Rightarrow\) (b). Suppose there exist a closed set \(A\) and a filter base \(\mathcal{B}\) on \(P(A) - \mathfrak{I}\) which does not r-accumulate to any \(a \in A\). Then for each \(a \in A\), there exists an open set \(U(a)\) containing \(a\) and a \(B(a) \in \mathcal{B}\) such that \(B(a) \cap \text{cl}(U(a)) = \emptyset\). Then \(\{U(a) : a \in A\}\) is an open cover of the closed set \(A\). By (a), there exists a finite subcollection \(\{U(a_i) : i = 1, 2, 3, \ldots, n\}\) such that \(A - \bigcup_{i=1}^{n} \text{cl}(U(a_i)) \in \mathfrak{I}\). If \(B \in \mathcal{B}\) is such that \(B \subset \bigcap_{i=1}^{n} B(a_i)\), then \(B \cap (A - \bigcup_{i=1}^{n} \text{cl}(U(a_i))) \in \mathfrak{I}\), that is, \(B - \bigcup_{i=1}^{n} \text{cl}(U(a_i)) \in \mathfrak{I}\). But the later set is just \(B\), because \(B \subset B(a_i)\) and \(B(a_i) \cap \text{cl}(U(a_i)) = \emptyset\) for each \(i\). However, \(B \in \mathfrak{I}\) is a contradiction, because \(B \in \mathfrak{B}\) and \(\mathfrak{B} \subset P(A) - \mathfrak{I}\).

(b) \(\Leftrightarrow\) (c). This follows in view of parts (a), (b), and (c) of [4, Theorem 1].

(b) \(\Rightarrow\) (a). If possible, let \(X\) be not \(C(\mathfrak{I})\)-compact. Then by Theorem 3.6(f), there exist a closed set \(A\) and a collection \(\mathcal{F}\) of regular closed sets with the property that for every finite subcollection \(\{F_1, F_2, F_3, \ldots, F_n\}\), \(\bigcap_{i=1}^{n} \text{int}(F_i) \cap A \notin \mathfrak{I}\), but \(\bigcap\{F : F \in \mathcal{F}\} \cap A = \emptyset\). Now the collection of sets of the form \(\bigcap_{i=1}^{n} \text{int}(F_i) \cap A\) for all possible finite subfamilies \(\{F_1, F_2, F_3, \ldots, F_n\}\) of \(\mathcal{F}\) forms a filter base on \(P(A) - \mathfrak{I}\). By (b), this filter base r-accumulates to some \(a \in A\), that is, for each open set \(U(a)\) containing \(a\) and for each \(F \in \mathcal{F}\), \(\text{cl}(U(a)) \cap (\text{int}(F) \cap A) \neq \emptyset\). However, \(a \in \mathcal{A}\) and \(\mathcal{A} \cap \{F : F \in \mathcal{F}\} = \emptyset\) imply that there is some \(F = F(a) \in \mathcal{F}\) such that \(a \notin F(a)\). Then \(X \setminus F(a)\) is an open set containing \(a\) such that \(\text{cl}(X \setminus F(a)) \cap (\text{int}(F(a)) \cap A) = \emptyset\). This is a contradiction.

(b) \(\Leftrightarrow\) (d). This follows using standard arguments about nets and filters. \(\square\)

If in the above theorem, \(A\) is replaced by the whole space \(X\), we get the characterizations of \((\mathfrak{I})\) QHC spaces. If in addition we consider completely codense ideal \(\mathfrak{I}\), we get the characterizations of quasi-\(H\)-closed spaces.

### 4. \(C(\mathfrak{I})\)-compact spaces and functions

A function \(f : (X, \tau) \rightarrow (Y, \varsigma)\) is said to be \(\theta\)-continuous [2] at a point \(x \in X\) if for every open set \(V\) of \(Y\) containing \(f(x)\), there exists an open set \(U\) of \(X\) containing \(x\) such that \(f(\text{cl}(U)) \subseteq \text{cl}(V)\). A function \(f : (X, \tau) \rightarrow (Y, \varsigma)\) is said to be \(\theta\)-continuous if \(f\) is \(\theta\)-continuous for every \(x \in X\). The concept of \(\theta\)-continuity is weaker than that of continuity. An important property of \(C\)-compact spaces is that a continuous function from a \(C\)-compact space to a Hausdorff space is closed. We prove the following more general results.

**Theorem 4.1.** Let \(f : (X, \tau, \mathfrak{I}) \rightarrow (Y, \varsigma, \Theta)\) be a \(\theta\)-continuous function, \((X, \tau, \mathfrak{I})\) \(C(\mathfrak{I})\)-compact, \((Y, \varsigma, \Theta)\) Hausdorff, and \(f(\mathfrak{I}) \subseteq \Theta\). Then \(f(A)\) is \(\varsigma^{*}(\Theta)\)-closed for each closed set \(A\) of \(X\).
Proof. Let $A$ be any closed set in $X$ and $a \not\in f(A)$. For each $x \in A$, there exists a $\varsigma$-open set $V_x$ containing $y = f(x)$ such that $a \not\in \text{cl}(V_x)$. Now because $f$ is $\theta$-continuous, there exists an open set $U_x$ containing $x$ such that $f(\text{cl}(U_x)) \subseteq \text{cl}(V_y)$. The family $\{U_x : x \in A\}$ is an open cover of $A$. Therefore, there exists a finite subfamily $\{U_{x_i} : i = 1, 2, \ldots, n\}$ such that $A - \bigcup_{i=1}^{n} \text{cl}(U_{x_i}) \subseteq f(\emptyset)$. But then $f(A) - f(\bigcup_{i=1}^{n} \text{cl}(U_{x_i})) \subseteq f(\emptyset) \subseteq \emptyset$, that is, $f(A) - f(\bigcup_{i=1}^{n} \text{cl}(U_{x_i})) \subseteq f(\emptyset)$ because $f(\emptyset)$ is also an ideal. Hence $f(A) - \bigcup_{i=1}^{n} \text{cl}(V_{y_i}) \subseteq f(\emptyset)$. Now $a \not\in \text{cl}(V_{y_i})$ for any $i$ implies that $a \in Y - \bigcup_{i=1}^{n} \text{cl}(V_{y_i})$ which is open in $(Y, \varsigma)$ and $(Y - \bigcup_{i=1}^{n} \text{cl}(V_{y_i})) \cap f(A) = f(A) - \bigcup_{i=1}^{n} \text{cl}(V_{y_i}) \subseteq f(\emptyset)$. Hence $a \not\in f(A)^*(\sigma, \emptyset)$. Thus $(f(A))^*(\sigma, \emptyset) \not\subseteq f(A)$ and so $f(A)$ is $\varsigma^*(\emptyset)$-closed. \qed

**Corollary 4.2.** Let $f : (X, \tau, \mathcal{F}) - (Y, \varsigma, \emptyset)$ be a continuous function, $(X, \tau, \mathcal{F})$ $C(\mathcal{F})$-compact, $(Y, \varsigma)$ Hausdorff, and $f(\emptyset) \subseteq \emptyset$. Then $f(A)$ is $\varsigma^*(\emptyset)$-closed for each closed set $A$ of $X$.

**Theorem 4.3.** Let $f : (X, \tau, \mathcal{F}) - (Y, \varsigma, \emptyset)$ be a continuous surjection, $(X, \tau, \mathcal{F})$ $C(\mathcal{F})$-compact, and $f(\emptyset) \subseteq \emptyset$. Then $(Y, \varsigma, \emptyset)$ is $C(\emptyset)$-compact.

**Proof.** Let $A$ be any closed subset of $(Y, \varsigma)$ and $\{V_a : a \in \Lambda\}$ any open cover of $A$ by open sets in $Y$. Then $\{f^{-1}(V_a) : a \in \Lambda\}$ is an open cover of $f^{-1}(A)$ which is closed in $X$. Hence, by the hypothesis, there exists a finite subcollection $\{f^{-1}(V_a) : i = 1, 2, \ldots, n\}$ such that $f^{-1}(A) - \bigcup_{i=1}^{n} \text{cl}(f^{-1}(V_a)) \subseteq \emptyset$. Since $f$ is continuous, $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}(B))$ for every subset $B$ of $Y$. Hence we have $f^{-1}(A) - \bigcup_{i=1}^{n} \text{cl}(f^{-1}(V_a)) = f^{-1}(A) - \bigcup_{i=1}^{n} \text{cl}(V_a) \subseteq f(\emptyset)$. Since $f$ is surjective, $A - \bigcup_{i=1}^{n} \text{cl}(V_a) \subseteq f(\emptyset) \subseteq \emptyset$. Hence $Y$ is $C(\emptyset)$-compact. \qed

**Theorem 4.4.** If the product space $\Pi X_a$ of nonempty family of topological spaces $(X_a, \tau_a)$ is $C(\mathcal{F})$-compact, then each $(X_a, \tau_a)$ is $C(\rho_a(\mathcal{F}))$-compact, where $\rho_a$ is the projection map and $\mathcal{F}$ is an ideal on $\Pi X_a$.

**Proof.** This follows from Theorem 4.3. \qed

### 5. $C(\mathcal{F})$-compact spaces and subspaces

In this section, we introduce three types of $C(\mathcal{F})$-compact subsets and use them to obtain new characterizations of $C(\mathcal{F})$-compact spaces and a characterization of maximal $C(\mathcal{F})$-compact spaces.

**Definition 5.1.** Let $(X, \tau)$ be a space and $\mathcal{F}$ an ideal on $X$. A subset $Y$ of $X$ is said to be $C(\mathcal{F})$-compact if the subspace $(Y, \tau_Y)$ is $C(\mathcal{F})$-compact.

Some useful results about such subspaces are contained in the following theorem. The proofs are easy to establish.

**Theorem 5.2.** Let $(X, \tau)$ be a space and $\mathcal{F}$ an ideal on $X$. Then

(a) a subspace $Y$ is $C(\mathcal{F})$-compact if and only if it is $C(\mathcal{F}_Y)$-compact;
(b) a clopen subspace of a $C(\mathcal{F})$-compact space is $C(\mathcal{F})$-compact;
(c) if $Y$ is a regular closed subset of a $C(\mathcal{F})$-compact space $(X, \tau, \mathcal{F})$ and $\mathcal{F}$ is codense, then $(Y, \tau_Y)$ is quasi-$H$-closed;
(d) a finite union of $C(\mathcal{F})$-compact subspaces of $X$ is $C(\mathcal{F})$-compact.
Definition 5.3. A subset $Y$ of $(X, \tau)$ is said to be $C(\mathcal{F})$-compact relative to $\tau$ if every $\tau$-open cover of every relatively closed subset $A$ of $Y$ has a finite subfamily whose $\tau$-closures cover $A$ except a set in $\mathcal{F}$.

Some useful properties of such spaces are contained in the following.

Theorem 5.4. Let $(X, \tau)$ be a space and $\mathcal{F}$ an ideal on $X$. Then the following hold.

(a) A closed subspace of a $C(\mathcal{F})$-compact relative to $\tau$ subspace of $(X, \tau)$ is $C(\mathcal{F})$-compact relative to $\tau$.

(b) If $(X, \tau)$ is Hausdorff and $Y$ is $C(\mathcal{F})$-compact relative to $\tau$, then $Y$ is $\tau^*(\mathcal{F})$-closed.

(c) If $Y$ is a $C(\mathcal{F})$-compact relative to $\tau$ subspace of $(X, \tau)$ and $f : (X, \tau) - (Z, \zeta)$ is a continuous bijection, then $f(Y)$ is $C(f(\mathcal{F}))$-compact relative to $\zeta$.

(d) $C(\mathcal{F})$-compactness relative to $\tau$ is contractive.

The following characterization of $C(\mathcal{F})$-compact spaces is obtained using $C(\mathcal{F})$-compact relative to $\tau$ subspaces. The proof is easy.

Theorem 5.5. A space $(X, \tau)$ with an ideal $\mathcal{F}$ is $C(\mathcal{F})$-compact if and only if every proper closed subset of $X$ is $C(\mathcal{F})$-compact relative to $\tau$.

Definition 5.6. A subset $Y$ of a space $(X, \tau)$ is said to be closure $C(\mathcal{F})$-compact if for every $\tau$-closed subset $K$ of $Y$ and every $\tau$-open cover $\mathcal{U}$ of $\text{cl}(K)$, there is a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that $K - \bigcup_{i=1}^{n} \text{cl}_{\tau}(U_i \cap Y) \in \mathcal{F}$.

Example 5.7. Since closed subsets of $C(\mathcal{F})$-compact spaces are not necessarily $C(\mathcal{F})$ QHC, a space $(X, \tau)$ which is $C(\mathcal{F})$-compact relative to $\tau$ may fail to be closure $C(\mathcal{F})$-compact. Moreover, $[0, 1]$ as a subspace of $[0, 1]$ is closure $C(\mathcal{F})$-compact with $\mathcal{F} = \{\phi\}$, but not $C(\mathcal{F})$-compact relative to the usual topology. Thus the concepts of $C(\mathcal{F})$-compact relative to $\tau$ and closure $C(\mathcal{F})$-compact are independent concepts.

We now have the following characterization of $C(\mathcal{F})$-compact spaces.

Theorem 5.8. A space $(X, \tau)$ is $C(\mathcal{F})$-compact for an ideal $\mathcal{F}$ on $X$ if and only if every open subset of $X$ is closure $C(\mathcal{F})$-compact.

Proof. Let $(X, \tau)$ be $C(\mathcal{F})$-compact and $Y$ an open subset of $X$. Let $K$ be a $\tau_Y$-closed subset of $Y$, and let $\mathcal{U}$ be a $\tau$-open cover of $\text{cl}(K)$. Then there exists a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U}$ such that $\text{cl}(K) - \bigcup_{i=1}^{n} \text{cl}_{\tau}(U_i) \in \mathcal{F}$. Since $Y$ is open, therefore, $\text{cl}_Y(U \cap Y) = \text{cl}(U) \cap Y$ and so, by hereditary property of $\mathcal{F}$, $K - \bigcup_{i=1}^{n} \text{cl}_{\tau}(U_i \cap Y) \in \mathcal{F}$. Thus $Y$ is closure $C(\mathcal{F})$-compact.

Conversely, let all open subsets of $X$ be closure $C(\mathcal{F})$-compact. Let $K$ be a closed and $\mathcal{U}$ an open cover of $K$. Choose a $U_0 \in \mathcal{U}$. Then $Y = X - \text{cl}(U_0)$ is an open subset of $X$ and $K \cap Y$ is a $\tau_Y$-closed subset of $Y$. Moreover, $\mathcal{U} - \{U_0\}$ is an open cover of $\text{cl}(K \cap Y)$. By the hypothesis, there exists a finite subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of $\mathcal{U} - \{U_0\}$ such that $K \cap Y - \bigcup_{i=1}^{n} \text{cl}_{\tau}(U_i \cap Y) \in \mathcal{F}$. But then $K \cap Y - \bigcup_{i=1}^{n} \text{cl}(U_i) \in \mathcal{F}$ as $\text{cl}_{\tau}(U_i \cap Y) = \text{cl}(U_i) \cap Y$ and $\mathcal{F}$ is hereditary. Therefore, $K - \bigcup_{i=0}^{n} \text{cl}(U_i) \in \mathcal{F}$. Hence $(X, \tau)$ is $C(\mathcal{F})$-compact.
Finally, we obtain a characterization of a maximal $C(\mathcal{F})$-compact space. Recall that a space $(X, \tau)$ with property $P$ is said to be maximal $P$ if there is no topology $\sigma$ on $X$ which has property $P$ and is strictly finer than $\tau$. For a topological space $(X, \tau)$ and a subset $A$ of $X$, $\tau(A) = \{U \cup (V \cap A) : U, V \in \tau\}$ is a topology called simple extension [7] of $\tau$ by $A$. $\tau(A)$ is strictly finer than $\tau$ if and only if $A \notin \tau$.

**Theorem 5.9.** A topological space $(X, \tau)$ is maximal $C(\mathcal{F})$-compact if and only if for every subset $A$ of $X$ such that $A$ is closure $C(\mathcal{F})$-compact and $X - A$ is $C(\mathcal{F})$-compact relative to $\tau$, one has $A \in \tau$.

**Proof.** First we assume that $(X, \tau)$ is maximal $C(\mathcal{F})$-compact and that $A$ is a subset of $X$ satisfying the given conditions. First, we show that $(X, \tau(A))$ is $C(\mathcal{F})$-compact. Let $K$ be a $\tau(A)$-closed subset of $X$. Then $K = K_1 \cup (K_2 \cap (X - A))$, where $K_1$ and $K_2$ are $\tau$-closed sets. Let

$$\mathcal{U} = \{U_\alpha \cup (V_\alpha \cap A) : U_\alpha, V_\alpha \in \tau, \alpha \in \Delta\}$$

be a $\tau(A)$-open cover of $K$. Then $\nu = \{U_\alpha : \alpha \in \Delta\}$ is a $\tau$-open cover of $K \cap (X - A) = (K_1 \cup K_2) \cap (X - A)$. Since, by assumption, $X - A$ is $C(\mathcal{F})$-compact relative to $\tau$, we have a finite subcollection $\{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}\}$ of $\nu$ such that $K \cap (X - A) - \bigcup_{i=1}^n \text{cl}(U_{\alpha_i}) \in \mathcal{F}$. Since $\tau(A)$ is finer than $\tau$, this subcollection is $\tau(A)$-open and $K \cap (X - A) - \bigcup_{i=1}^n \text{cl}_{\tau(A)}(U_{\alpha_i}) \in \mathcal{F}$. Next, $\mathcal{W} = \{U_\alpha \cup V_\alpha : \alpha \in \Delta\}$ is a $\tau$-open cover of $\text{cl}(K \cap A) = \text{cl}(K_1 \cap A) = \text{cl}_{\tau(A)}(K_1 \cap A)$ and therefore by assumption on $A$, there exists a finite subcollection $\{U_{\beta_i} \cup V_{\beta_i} : i = 1, 2, \ldots, k\}$ of $\mathcal{W}$ such that

$$K_1 \cap A - \bigcup_{i=1}^k \text{cl}_{\tau(A)}[(U_{\beta_i} \cup V_{\beta_i}) \cap A] \in \mathcal{F}. \quad (5.2)$$

However, $\tau_A$, the restriction of $\tau$ to $A$, is nothing but $\tau(A) | A$, the restriction of $\tau(A)$ to $A$. Therefore,

$$K_1 \cap A - \bigcup_{i=1}^k \text{cl}_{\tau(A) | A}[(U_{\beta_i} \cup V_{\beta_i}) \cap A] \in \mathcal{F}. \quad (5.3)$$

Now $\{U_{\alpha_i} \cup (V_{\alpha_i} \cap A) : i = 1, 2, \ldots, n\} \cup \{U_{\beta_i} \cup (V_{\beta_i} \cap A) : i = 1, 2, \ldots, k\}$ is a finite $\tau(A)$ ($\mathcal{F}$) proximate cover of $K$ which is a subcover of $\mathcal{U}$. Thus the topology $\tau(A)$ on $X$ is also $C(\mathcal{F})$-compact. However, by the maximality of $\tau$, we have $\tau(A) = \tau$. But then $A \in \tau$ as desired.

Conversely, let $(X, \tau)$ be not maximal $C(\mathcal{F})$-compact. Then there is a $C(\mathcal{F})$-compact topology $\sigma$ on $X$ which is strictly finer than $\tau$. Let $A \in \sigma - \tau$. Then $A$ is $\sigma$-closure $C(\mathcal{F})$-compact by Theorem 5.8. Since the property of closure $C(\mathcal{F})$-compact is carried over to coarser topologies, $A$ is $\tau$-closure $C(\mathcal{F})$-compact. Also $X - A$ is $C(\mathcal{F})$-compact relative to $\sigma$ and hence $C(\mathcal{F})$-compact relative to $\tau$. By the hypothesis, then $A \in \tau$, a contradiction.

**Remark 5.10.** The readers can generalize the above concepts in bitopological spaces to unify various types of compactness.
12 \( C \)-compactness modulo an ideal

References


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