The main purpose of this paper is to characterize the classes of matrices \( \text{ces}[(p),(q)],c^\sigma \) and \( \text{ces}[(p),(q)],l^\sigma_{\infty} \), where \( c^\sigma \) is the space of all bounded sequences all of whose \( \sigma \)-means are equal, \( l^\sigma_{\infty} \) is the space of \( \sigma \)-bounded sequences, and \( \text{ces}[(p),(q)] \) is the generalized Cesàro sequence space.

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1. Introduction

Let \( \omega \) be the space of all sequences, real or complex, and let \( l_{\infty} \) and \( c \), respectively, be the Banach spaces of bounded and convergent sequences \( x = (x_n) \) with norm \( \|x\| = \sup_{k \in \mathbb{Z}} |x_k| \). Let \( \sigma \) be a mapping of the set of positive integers into itself. A continuous linear functional \( \phi \) on \( l_{\infty} \) is said to be an invariant mean or a \( \sigma \)-mean if and only if (i) \( \phi(x) \geq 0 \), when the sequence \( x = (x_n) \) has \( x_n \geq 0 \) for each \( n \); (ii) \( \phi(e) = 1 \), where \( e = (1,1,1,...) \); and (iii) \( \phi((x_{\sigma(n)})) = \phi(x), x \in l_{\infty} \).

For certain kinds of mappings, every \( \sigma \)-mean extends the limit functional \( \phi \) on \( c \) in the sense that \( \phi(x) = \lim x \) for \( x \in c \) (see [2, 15]). Consequently, \( c \subset c^\sigma \), where \( c^\sigma \) is the set of bounded sequences, all of whose invariant means are equal (see [1, 9, 10]). When \( \sigma \) is translation, the \( \sigma \)-means are classical Banach limits on \( l_{\infty} \) (see [2]) and \( c^\sigma \) is the set of almost convergent sequences \( \hat{c} \) (see [7]). Almost convergence for double sequences was introduced and studied by Móricz and Rhoades [8] and further by Mursaleen and Savaş [13], Mursaleen and Edely [12], and Mursaleen [11].

If \( x = (x_n) \), write \( Tx = (Tx_n) = (x_{\sigma(n)}) \), then

\[
c^\sigma = \left\{ x \in l_{\infty} : \lim_{m \to \infty} t_{m,n}(x) = L, \text{ uniformly in } n, L = \sigma - \lim x \right\},
\]

where

\[
t_{m,n}(x) = \frac{1}{m+1} \sum_{j=0}^{m} T^j x_n \quad \text{with} \quad T^j x_n = x_{\sigma^j(n)}, \quad t_{-1,n}(x) = 0.
\]

We define \( l^\sigma_{\infty} \) the space of \( \sigma \)-bounded sequences (Ahmad et al. [2]) in the following way.
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Let \(x_n = z_0 + z_1 + z_2 + \cdots + z_n\) and

\[
I_\sigma^\infty = \left\{ z \in \omega : \sup_{m,n} |\psi_{m,n}(z)| < \infty \right\},
\]

(1.3)

where

\[
\psi_{m,n}(z) = t_{m,n}(x) - t_{m-1,n}(x)
\]

\[
= \frac{1}{m(m+1)} \sum_{j=1}^{m} \sum_{i=h_{j-1}+1}^{h_{j}} z_i, \quad h_j = \sigma_j(n).
\]

(1.4)

If \(\sigma(n) = (n+1)\), then \(I_\sigma^\infty\) is the set of almost bounded sequences \(\hat{l}_\infty\) (see [14]).

Let \(A = (a_{nk})\) be an infinite matrix of complex numbers \(a_{nk}\) \((n,k = 1,2,\ldots)\) and \(X, Y\) two subsets of \(\omega\). We say that the matrix \(A\) defines a matrix transformation from \(X\) into \(Y\) if for every sequence \(x = (x_k) \in X\) the sequence \(A(x) = (A_n(x)) \in Y\), where \(A_n(x) = \sum_k a_{nk} x_k\) converges for each \(n\). We denote the class of matrix transformations from \(X\) into \(Y\) by \((X, Y)\).

The main purpose of this paper is to characterize the classes \((\text{ces}[(p),(q)], c^\sigma)\) and \((\text{ces}[(p),(q)], I_\sigma^\infty)\) and deduce some known and unknown interesting results as corollaries.

The classes \((\text{ces}[(p),(q)], c^\sigma)\) and \((\text{ces}[(p),(q)], I_\sigma^\infty)\) are due to Khan and Rahman [4].

If \(\{q_n\}\) is a sequence of positive real numbers, then for \(p = (p_r)\) with \(\inf p_r > 0\), we define the space \(\text{ces}[(p),(q)]\) by

\[
\text{ces}[(p),(q)] = \left\{ x \in \omega : \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2^r}} \sum_{r} q_k |x_k| \right)^{p_r} < \infty \right\},
\]

(1.5)

where \(Q_{2^r} = q_{2^r} + q_{2^r+1} + \cdots + q_{2^{r+1}-1}\) and \(\sum_r\) denotes a sum over the range \(2^r \leq k < 2^{r+1}\).

Remark 1.1. If \(q_n = 1\) for all \(n\), then \(\text{ces}[(p),(q)]\) reduces to \(\text{ces}(p)\) studied by Lim [6]. Also, if \(p_n = p\) for all \(n\) and \(q_n = 1\) for all \(n\), then \(\text{ces}[(p),(q)]\) reduces to \(\text{ces}_p\) studied by Lim [5].

For any bounded sequence \(p\), the space \(\text{ces}[(p),(q)]\) is a paranormed space with the paranorm given by (see [4])

\[
g(x) = \left( \sum_{r=0}^{\infty} \left( \frac{1}{Q_{2^r}} \sum_{r} q_k |x_k| \right)^{p_r} \right)^{1/M}
\]

(1.6)

if \(H = \sup_r p_r < \infty\) and \(M = \max(1,H)\).
2. Sequence-to-sequence transformations

In this section, we characterize the classes \((\text{ces}[(p),(q)], c^\sigma)\) and \((\text{ces}[(p),(q)], l_\infty^p)\).

We write \(a(n,k)\) to denote the elements \(a_{nk}\) of the matrix \(A\), and for all integers \(n,m \geq 1\), we write

\[
t_{mn}(Ax) = \frac{Ax_n + TAx_n + \cdots + T^mAx_n}{m + 1} = \sum_k t(n,k,m)x_k, \tag{2.1}
\]

where \(t(n,k,m) = 1/(m + 1)\sum_{j=0}^{m}a(\sigma^j(n),k)\).

We also define the spaces of \(\sigma\)-convergent series and \(\sigma\)-bounded series, respectively, as follows:

\[
c_{\sigma}^r = \left\{ x : \sum_{i=1}^{m} \frac{1}{i + 1} \sum_{j=0}^{i} x_{\sigma^j(n)} \right\} \text{ is convergent uniformly in } n, \text{ as } m \to \infty \right\}, \tag{2.2}
\]

\[
b_{\sigma}^r = \left\{ x : \sup_{n,m} \sum_{i=1}^{m} \frac{1}{i + 1} \sum_{j=0}^{i} x_{\sigma^j(n)} < \infty \right\}.
\]

If we take \(\sigma(n) = n + 1\), \(c_{\sigma}^r\) and \(b_{\sigma}^r\) reduce to \(\hat{c}_r\) and \(\hat{b}_r\), as defined below:

\[
\hat{c}_r = \left\{ x : \sum_{i=1}^{m} \frac{1}{i + 1} \sum_{j=0}^{i} x_{j+n} \right\} \text{ is convergent uniformly in } n, \text{ as } m \to \infty \right\}, \tag{2.3}
\]

\[
\hat{b}_r = \left\{ x : \sup_{n,m} \sum_{i=1}^{m} \frac{1}{i + 1} \sum_{j=0}^{i} x_{j+n} < \infty \right\}.
\]

Now we prove the following theorem.

**Theorem 2.1.** Let \(1 < p_r \leq \sup_r p_r < \infty\). Then \(A \in (\text{ces}[(p),(q)], c^\sigma)\) if and only if

(i) there exists an integer \(E > 1\) such that for all \(n,\)

\[
U(E) = \sup_m \sum_{r=0}^{\infty} \left( Q_{2^r} \max_{r} \left( \frac{|t(n,k,m)|}{q_k} \right) \right)^{t_r} E^{-t_r} < \infty, \tag{2.4}
\]

where \(1/p_r + 1/t_r = 1, r = 0,1,2,\ldots,\) and \(\max_r\) means maximum over \(2^r \leq k < 2^{r+1}\);

(ii) \(a_{nk} = (a_{nk})_{n=1}^{\infty} \in c^\sigma\) for each \(k\), that is, \(\lim_m t(n,k,m) = u_k\) uniformly in \(n\), for each \(k\).

In this case, \(\sigma\)-limit of \(Ax\) is \(\sum_{k=1}^{\infty} u_kx_k\).

**Proof**

**Necessity.** Suppose that \(A \in (\text{ces}[(p),(q)], c^\sigma)\). Now \(\sum_{k=1}^{\infty} t(n,k,m)x_k\) exists for each \(m\) and \(n\) and \(x \in \text{ces}[(p),(q)]\), whence \(\{t(n,k,m)\}_{k} \in \text{ces}^*[(p),(q)]\) for each \(m\) and \(n\) (see F. M. Khan and M. A. Khan [3] for Köthe-Toeplitz and continuous duals of \(\text{ces}[(p),(q)]\)).
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Therefore, it follows that each \( \{ f_{m,n} \}_m \) defined by

\[
f_{m,n}(x) = t_{m,n}(Ax)
\]

is an element of \( \text{ces}^*[(p),(q)] \). Since \( \text{ces}[(p),(q)] \) is complete and further for each \( n \), \( \sup_m |t_{m,n}(Ax)| < \infty \) on \( \text{ces}[(p),(q)] \). Now arguing with the uniform boundedness principle, we have condition (i). Since \( e_k \in \text{ces}[(p),(q)] \), condition (ii) follows.

**Sufficiency.** Suppose that the conditions hold. Fix \( n \in \mathbb{N} \). For every integer \( s \geq 1 \), from (i) we have

\[
\sum_{r=0}^{s} \left( Q_{2r} \max_{r}(q_k^{-1} | t(n,k,m) | ) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r}(q_k^{-1} | t(n,k,m) | ) \right)^{t_r} E^{-t_r}.
\]

(2.6)

Now letting \( s \to \infty \), we obtain

\[
\lim_{m \to \infty} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r}(q_k^{-1} | t(n,k,m) | ) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r}(q_k^{-1} | t(n,k,m) | ) \right)^{t_r} E^{-t_r}.
\]

(2.7)

Therefore, from (ii) we have

\[
\sum_{r=0}^{\infty} \left( Q_{2r} \max_{r}(q_k^{-1} | u_k | ) \right)^{t_r} E^{-t_r} \leq \sup_m \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r}(q_k^{-1} | t(n,k,m) | ) \right)^{t_r} E^{-t_r} < \infty.
\]

(2.8)

Hence \( (u_k)_k \) and \( \{ t(n,k,m) \}_k \in \text{ces}^*[(p),(q)] \), therefore the series \( \sum_{k=1}^{\infty} t(n,k,m)x_k \) and \( \sum_{k=1}^{\infty} u_kx_k \) converge for each \( m \) and \( n \) and \( x \in \text{ces}[(p),(q)] \). For given \( \epsilon > 0 \) and \( x \in \text{ces}[(p),(q)] \), choose \( s \) such that

\[
\left( \sum_{r=s+1}^{\infty} \left( \frac{1}{Q_{2r}} \sum_{r} q_k | x_k | \right)^{p_r} \right)^{1/M} < \epsilon.
\]

(2.9)

Since (ii) holds, there exists \( m_0 \) such that

\[
\left| \sum_{k=1}^{s} t(n,k,m) - u_k \right| < \epsilon \quad \forall m > m_0.
\]

(2.10)

Since (i) holds, it follows that

\[
\left| \sum_{k=s+1}^{\infty} t(n,k,m) - u_k \right| \text{ is arbitrary small.}
\]

(2.11)
Therefore,

\[
\lim_{m \to \infty} \sum_{k=1}^{\infty} t(n,k,m)x_k = \sum_{k=1}^{\infty} u_kx_k, \quad \text{uniformly in } n. \tag{2.12}
\]

This completes the proof. \(\square\)

**Remark 2.1.** For different choices of \(p, q,\) and \(\sigma,\) we can deduce many corollaries from the above theorem to characterize the matrix classes, for example, \((\text{ces}(p), c^\sigma),\) \((\text{ces}_p, c^\sigma),\) \((\text{ces}_p(q), c^\sigma),\) \((\text{ces}([p), (q)], \hat{c}),\) and so forth. The class \((\text{ces}(p), \hat{c})\) was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from our theorem by taking \(q_n = 1\) for all \(n\) and \(\sigma(n) = n + 1.\)

We write (see [2])

\[
x_0 = z_0 + z_1 + \cdots + z_n,
\]

\[
\psi_{m,n}(Az) = \sum_k \alpha(n,k,m)z_k, \tag{2.13}
\]

where

\[
\alpha(n,k,m) = \frac{1}{m(m+1)} \sum_{j=1}^{m} \left[ \sum_{i=h_j-1+1}^{h_j} a_{ik} \right], \quad h_j = \sigma^j(n). \tag{2.14}
\]

Now we prove the following theorem.

**Theorem 2.2.** Let \(1 < p_r \leq \sup_r p_r < \infty.\) Then \(A \in (\text{ces}([p), (q)], l^\sigma_\infty)\) if and only if

\[
\sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_k^{-1} | \alpha(n,k,m) |) \right)^{t_r} E^{-t_r} < \infty, \tag{2.15}
\]

where \(E\) is an integer greater than \(1\) and \(1/p_r + 1/t_r = 1, r = 0, 1, 2,\ldots.\)

**Proof**

**Necessity.** Suppose that \(A \in (\text{ces}([p), (q)], l^\sigma_\infty).\) Now \(\sum_{k=1}^{\infty} \alpha(n,k,m)z_k\) exists for each \(m\) and \(n\) and \(z \in \text{ces}([p), (q)],\) whence \(\{\alpha(n,k,m)\}_{k} \in \text{ces}^*([p), (q)]\) for each \(m\) and \(n.\) Therefore, it follows that \(\{f_{m,n}\} \) defined by

\[
f_{m,n}(x) = \psi_{m,n}(Az) \tag{2.16}
\]

is an element of \(\text{ces}^*([p), (q)].\) Since \(\text{ces}([p), (q)]\) is complete and further \(\sup_{m,n} |\psi_{m,n}(Az)| < \infty\) on \(\text{ces}([p), (q)],\) so by arguing with uniform boundedness principle, we have the condition.
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Sufficiency. Suppose that condition (2.15) holds. Fix \( n \in \mathbb{N} \). For every integer \( s \geq 1 \) we have
\[
\sum_{r=0}^{s} \left( Q_{2r} \max_{r} (q_k^{-1} | a(n,k,m) |) \right)^{t_r} E^{-s} \leq \sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_k^{-1} | a(n,k,m) |) \right)^{t_r} E^{-s}.
\]
(2.17)

So
\[
\lim_{s \to \infty} \sum_{r=0}^{s} \left( Q_{2r} \max_{r} (q_k^{-1} | a(n,k,m) |) \right)^{t_r} E^{-s} \leq \sup_{m,n} \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} (q_k^{-1} | a(n,k,m) |) \right)^{t_r} E^{-s} < \infty.
\]
(2.18)

Hence \( \{a(n,k,m)\} \in \text{ces}^*[((p),(q))] \). Therefore, the series \( \sum_{k=1}^{\infty} a(n,k,m)z_k \) converges for each \( m \) and \( n \) and \( z \in \text{ces}((p),(q)) \).

This completes the proof. \( \square \)

Remark 2.2. The matrix class \((\text{ces}(p),\text{ces}(q))\), was characterized by F. M. Khan and M. A. Khan [3] which we can obtain directly from the above theorem by letting \( q_n = 1 \) for all \( n \) and \( \sigma(n) = n + 1 \). Besides, we can further deduce many corollaries for different choices of \( p, q, \) and \( \sigma \).

3. Sequence-to-series transformations

For all integers \( m,n \geq 1 \), we write
\[
i^*_mn(Ax) = \sum_{i=1}^{m} t_{in}(Ax) = \sum_{k=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n),k)x_k = \sum_{k} t^*(m,n,k)x_k,
\]
(3.1)

where
\[
t^*(m,n,k) = \sum_{i=1}^{m} \frac{1}{i+1} \sum_{j=0}^{i} a(\sigma^j(n),k).
\]
(3.2)

Theorem 3.1. Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (\text{ces}((p),(q)),\text{ces}) \) if and only if

(i) there exists an integer \( E > 1 \) such that for all \( n \),
\[
U(E) = \sup_m \sum_{r=0}^{\infty} \left( Q_{2r} \max_{r} \left( \frac{|t^*(n,k,m)|}{q_k} \right) \right)^{t_r} E^{-r} < \infty,
\]
(3.3)

where \( 1/p_r + 1/t_r = 1, r = 0,1,2,\ldots \), and \( \max_r \) means maximum over \( 2r \leq k \leq 2r+1 \);
(ii) \( a(k) = \{a_{nk}\}_{n=1}^\infty \in c_0^\sigma \) for each \( k \), that is, \( \lim_{m} t^*(n,k,m) = u_k \) uniformly in \( n \), for each \( k \).

In this case, the \( \sigma \)-limit of \( Ax \) is \( \sum_{k=1}^\infty u_k x_k \).

**Theorem 3.2.** Let \( 1 < p_r \leq \sup_r p_r < \infty \). Then \( A \in (ces[(p),(q)],b_\sigma) \) if and only if

\[
\sup_{m,n} \sum_{r=0}^\infty \left( Q_{2^r} \max_{i} \left( t^*(n,k,m) \right) \right)^{t_i} E^{-t_i} < \infty,
\]

where \( E \) is an integer greater than 1 and \( 1/p_r + 1/t_r = 1, r = 0,1,2,\ldots \).

Proofs of Theorems 3.1 and 3.2 are similar to those of Theorems 2.1 and 2.2, respectively.

**Remark 3.1.** If \( \sigma \) is translation, then Theorems 3.1 and 3.2 give the characterization for the classes \( (ces[(p),(q)],\hat{c}_\sigma) \) and \( (ces[(p),(q)],\hat{b}_\sigma) \). As Remarks 2.1 and 2.2, for different choices of \( p, q, \) and \( \sigma \), we can deduce many corollaries.

**References**


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