The formal Laplace-Borel transform of an analytic integral operator, known as a Fliess operator, is defined and developed. Then, in conjunction with the composition product over formal power series, the formal Laplace-Borel transform is shown to provide an isomorphism between the semigroup of all Fliess operators under operator composition and the semigroup of all locally convergent formal power series under the composition product. Finally, the formal Laplace-Borel transform is applied in a systems theory setting to explicitly derive the relationship between the formal Laplace transform of the input and output functions of a Fliess operator. This gives a compact interpretation of the operational calculus of Fliess for computing the output response of an analytic nonlinear system.

Copyright © 2006 Hindawi Publishing Corporation. All rights reserved.

1. Introduction

Let \( u : \mathbb{R} \to \mathbb{R} \) be a function which is real analytic at a point \( t_0 \in \mathbb{R} \). Its Taylor series expansion is denoted by

\[
    u(t) = \sum_{n=0}^{\infty} c_u(n) \frac{(t-t_0)^n}{n!}.
\]  

Within its radius of convergence, \( u \) is completely characterized by the sequence of coefficients \( \{c_u(n)\}_{n=0}^{\infty} \). From this, one can construct a formal power series representation of \( u \) by introducing an abstract symbol set \( X_0 = \{x_0\} \), namely,

\[
    c_u = \sum_{n=0}^{\infty} c_u(n)x_0^n.
\]  

The formal Laplace transform in this setting is the mapping

\[
    \mathcal{L}_f : u \mapsto c_u.
\]
Its inverse is the formal Borel transform. When \( u \) is entire, its one-sided integral Laplace transform is often well defined. Setting \( t_0 = 0 \), it can be written in the form

\[
\mathcal{L}[u](s) = \sum_{n=0}^{\infty} c_n(n) \int_0^\infty t^n e^{-st} dt
\]

\[
= s^{-1} \sum_{n=0}^{\infty} c_n(n)(s^{-1})^n,
\]

using the Laplace transformation pair

\[
t^n \Leftrightarrow n!(s^{-1})^{n+1}, \quad n \geq 0.
\]

Clearly, the two transforms \( \mathcal{L}_f \) and \( \mathcal{L} \) are related via

\[
\mathcal{L}[u](s) = x_0 \mathcal{L}_f[u] \big|_{x_0 \rightarrow s^{-1}}.
\]
For a measurable function \( u : [a, b] \to \mathbb{R}^m \), define \( \| u \|_p = \max \{ \| u_i \|_p : 1 \leq i \leq m \} \), where \( \| u_i \|_p \) is the usual \( L_p \)-norm for a measurable real-valued function, \( u_i \), defined on \([a, b]\) when \( p \in [1, \infty) \), and \( \| u \|_\infty = \sup_{t \in [a, b]} \{ |u_i(t)| : 1 \leq i \leq m \} \). For any fixed \( p \in [1, \infty) \), let \( L^m_p[a, b] \) denote the set of all measurable functions defined on \([a, b]\) having a finite \( \| \cdot \|_p \) norm and \( B^m_p(R)[a, b] := \{ u \in L^m_p[a, b] : \| u \|_p \leq R \} \). With \( t_0, T \in \mathbb{R} \) fixed, and \( T > 0 \), define inductively for each \( \eta = x_i \eta^i \in X^* \) the mapping \( E_\eta : L^m_p[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T] \) by

\[
E_\eta[u](t, t_0) := \int_{t_0}^t u_\eta(\tau)E_\eta[u](\tau, t_0) \, d\tau,
\]

where \( E_\varnothing \equiv 1 \) and \( u_0 \equiv 1 \). The input-output operator corresponding to \( c \) is the Fliess operator

\[
F_c[u](t) := \sum_{\eta \in X^*} (c, \eta)E_\eta[u](t, t_0).
\]

All Volterra operators with analytic kernels, for example, are Fliess operators. When there exist real numbers \( K, M > 0 \) such that

\[
| (c, \eta) | := \max \{ | (c, \eta)_j | : 0 \leq j \leq \ell \} \leq KM^{|\eta|}|\eta|!,
\]

the formal power series \( c \) is said to be \textit{locally convergent}. (Here \( |\eta| \) denotes the number of symbols in \( \eta \in X^* \).) The subset of all locally convergent series is denoted by \( \mathbb{R}^{\ell}_{LC}(\langle X \rangle) \). When \( c \in \mathbb{R}^{\ell}_{LC}(\langle X \rangle) \), it is known that \( F_c \) constitutes a well-defined operator from \( B^m_p(R)[t_0, t_0 + T] \) into \( B^m_q(S)[t_0, t_0 + T] \) for sufficiently small \( R, S, T > 0 \), where the numbers \( p, q \in [1, \infty] \) are conjugate exponents, that is, \( 1/p + 1/q = 1 \) with \( (1, \infty) \) being a conjugate pair by convention [17]. Therefore, the specific operator class of interest in this paper is the set of Fliess operators \( \mathcal{F} := \{ F_c : c \in \mathbb{R}^{\ell}_{LC}(\langle X \rangle) \} \). When \( \ell = m \), this set forms a semigroup under operator composition, as does the set \( \mathbb{R}^{\ell}_{LC}(\langle X \rangle) \) under the composition product [4, 20]. It will be shown here that the formal Laplace-Borel transform provides an isomorphism between these two semigroups.

The paper is organized as follows. In Section 2, the notion of a formal Laplace-Borel transform of a Fliess operator is defined. Then its basic properties are developed and a set of examples is given. In Section 3, the composition product is introduced and its relationship to the formal Laplace-Borel transform is described. Using the concepts developed in Section 3, the formal Laplace-Borel transform is shown to provide a compact interpretation of the operational calculus of Fliess in Section 4. Examples are given to illustrate the computation of the output response of an analytic system. Section 5 concludes the paper with a brief summary of the main results.

2. The formal Laplace-Borel transform of a Fliess operator

Consider a causal linear integral operator

\[
y(t) = \int_{t_0}^t h(t - \tau)u(\tau) \, d\tau,
\]

Y. Li and W. S. Gray 3
where the kernel function, \( h \), is analytic at \( t = 0 \). The operator is completely characterized by the Laplace transform of the kernel function, namely, \( H(s) := \mathcal{L}[h](s) = \sum_{k \geq 0} c_h(k)s^{-k} \), when it exists. Therefore, the usual definition (1.3) of the formal Laplace-Borel transform over a single symbol applies directly to this case. In the more general context of Fliess operators, a generalization is required. The following preliminaries are needed to ensure the new definition is well-posed.

**Theorem 2.1** [24, Corollary 2.2.4]. Suppose \( c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \). If \( F_c = F_d \) on \( B_m^m(1)[t_0, t_0 + T] \) for some finite \( T > 0 \), then \( c = d \).

Given the nested nature of the set of spaces \( L_p[t_0, t_0 + T] \) for \( p = 1, 2, \ldots, \infty \), the following result is immediate.

**Corollary 2.2.** Suppose \( c, d \in \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \). If \( F_c = F_d \) on \( B_p^p(R)[t_0, t_0 + T] \) for some \( p \in \{1, 2, \ldots, \infty\} \) and real numbers \( R, T > 0 \), then \( c = d \).

In light of this uniqueness property, there exists a one-to-one mapping between the set of well-defined Fliess operators, \( \mathcal{F} \), and the set of locally convergent formal power series, \( \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \). This guarantees that the following definition is well-posed.

**Definition 2.3.** Let \( X = \{x_0, x_1, \ldots, x_m\} \). The formal Laplace transform is defined as

\[
\mathcal{L}_f : \mathcal{F} \rightarrow \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle
\]

\[
: F_c \mapsto c.
\]

The corresponding inverse transform, the formal Borel transform, is

\[
\mathcal{B}_f : \mathbb{R}^{\ell}_{LC}\langle\langle X\rangle\rangle \rightarrow \mathcal{F}
\]

\[
: c \mapsto F_c.
\]

Note that when \( m = 0 \), this definition is consistent with that given in (1.3) in the sense that a fixed function \( u \) can be represented as the constant operator \( F_{cu} \), that is, \( u = F_{cu}[v] \) for all \( v \in B_p^p(R)[t_0, t_0 + T] \) and \( \mathcal{L}_f[u] = c_u = \mathcal{L}_f[F_{cu}] \).

It is next shown that many of the familiar properties of the integral Laplace transform also have counterparts in the present context. To facilitate the analysis, two concepts are needed.

**Definition 2.4.** For any \( x_i \in X \), the left-shift operator, \( x_i^{-1} : \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \rightarrow \mathbb{R}^{\ell}\langle\langle X\rangle\rangle \), is defined as

\[
x_i^{-1}(c) = \sum_{\eta \in X^*} (c, \eta)x_i^{-1}(\eta),
\]

where

\[
x_i^{-1}(\eta) = \begin{cases} 
\eta' : & \eta = x_i\eta', \quad \eta' \in X^*, \\
0 : & \text{otherwise}.
\end{cases}
\]
Definition 2.5. A Dirac series, \( \delta_i \), is a generalized series with the defining property that \( F_{\delta_i}[u] = u_i(t) \) for any \( 1 \leq i \leq m \).

The main result concerning elementary properties of the formal Laplace-Borel transform is stated below.

Theorem 2.6. Given any \( c, d \in \mathbb{R}_L^\ell \langle \langle X \rangle \rangle \) and scalars \( \alpha, \beta \in \mathbb{R} \), the following identities hold.

1. Linearity:
   \[
   \mathcal{L}_f[\alpha FC + \beta FD] = \alpha \mathcal{L}_f[Fc] + \beta \mathcal{L}_f[Fd],
   \]
   \[
   \mathcal{B}_f[\alpha c + \beta d] = \alpha \mathcal{B}_f[c] + \beta \mathcal{B}_f[d].
   \] (2.6)

2. Integration:
   \[
   \mathcal{L}_f[I^nFc] = x_0^n c,
   \]
   \[
   \mathcal{B}_f[x_0^n c] = I^nFc,
   \] (2.7)

   where \( I^n(\cdot) \) is the \( n \)th order integration operator so that
   \[
   I^nFc[u](t) = \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} Fc[u](\tau_n) d\tau_n \cdots d\tau_2 d\tau_1.
   \] (2.8)

3. Differentiation:
   \[
   \mathcal{L}_f[DFc] = x_0^{-1}(c) + \sum_{i=1}^{m} \delta_i \shuffle (x_i^{-1}(c)),
   \]
   \[
   \mathcal{B}_f\left[x_0^{-1}(c) + \sum_{i=1}^{m} \delta_i \shuffle (x_i^{-1}(c))\right] = DFc.
   \] (2.9)

If \( x_0^n \) is a left factor of \( c \), that is, \( c = x_0^n c' \) for some \( c' \in \mathbb{R}_L^\ell \langle \langle X \rangle \rangle \), then
   \[
   \mathcal{L}_f[D^nFc] = x_0^{-n}(c),
   \]
   \[
   \mathcal{B}_f[x_0^{-n}(c)] = D^nFc,
   \] (2.10)

   where \( D^n(\cdot) \) is the \( n \)th-order differentiation operator so that \( D^nFc[u](t) = d^nFc[u](t)/dt^n \), and \( \shuffle \) denotes the shuffle product (see, e.g., [21]).

4. Multiplication:
   \[
   \mathcal{L}_f[Fc \cdotFd] = \mathcal{L}_f[Fc] \shuffle \mathcal{L}_f[Fd],
   \]
   \[
   \mathcal{B}_f[c \shuffle d] = \mathcal{B}_f[c] \cdot \mathcal{B}_f[d].
   \] (2.11)

Proof. The properties of linearity and integration are trivial. The multiplication property follows from results in the literature concerning the shuffle product [22, 24]. Only the differentiation property remains to be justified. It is shown in [24] that the derivative of
The formal Laplace-Borel transform of Fliess operators

A Fliess operator is

\[
\frac{d}{dt} F_c[u](t) = F_{x_0^{-1}(c)}[u](t) + \sum_{i=1}^{m} u_i(t) F_{x_i^{-1}(c)}[u](t). \tag{2.12}
\]

Applying the formal Laplace transform to this equality gives the first pair of equations in part (3). Now if \( x_0 \) is a left factor of \( c \), then \( F_{x_i^{-1}(c)}[u](t) = 0 \) for \( i = 1, 2, \ldots, m \). In this case, \( dF_c[u](t)/dt = F_{x_0^{-1}(c)}[u](t) \). Proceeding inductively, the second pair of equations follows.  

The following definition is utilized in the examples which follow.

**Definition 2.7** [1]. Let \( c \in \mathbb{R} \langle \langle X \rangle \rangle \) be proper (i.e., \((c, \emptyset) = 0\)). Then the star operator applied to \( c \) is defined as

\[
c^* := \sum_{n \geq 0} c^n := (1 - c)^{-1}, \tag{2.13}
\]

where \( c^n \) denotes the catenation power.

Observe that when \( c \) is not proper, it is always possible to write \( c = (c, \emptyset)(1 - c') \). In which case, there exists a \( c^{-1} \in \mathbb{R} \langle \langle X \rangle \rangle \) such that under the catenation product \( cc^{-1} = 1 \) and \( c^{-1}c = 1 \). Specifically,

\[
c^{-1} = \frac{1}{(c, \emptyset)} (1 - c')^{-1} = \frac{1}{(c, \emptyset)} (c')^*. \tag{2.14}
\]

**Example 2.8.** Let \( X = \{x_0, x_1, x_2\} \) and \( F_c[u](t) = \exp(\int_0^t u_1(t) + u_2(t) dt) \). Observe that \( F_c[u] \) can be expanded as

\[
F_c[u](t) = \sum_{n \geq 0} \frac{1}{n!} \left( \int_0^t u_1(t) + u_2(t) dt \right)^n
= \sum_{n \geq 0} \int_0^t \left[ u_1(\tau_1) + u_2(\tau_1) \right] \int_0^{\tau_1} \left[ u_1(\tau_2) + u_2(\tau_2) \right] \cdots \int_0^{\tau_{n-1}} \left[ u_1(\tau_n) + u_2(\tau_n) \right] d\tau_n \cdots d\tau_2 d\tau_1. \tag{2.15}
\]

Therefore,

\[
\mathcal{L}_f[F_c] = \sum_{n \geq 0} (x_1 + x_2)^n = (x_1 + x_2)^*. \tag{2.16}
\]

This result can be viewed as an operator generalization of the integral transform pair

\[
e^t \iff (1 - s)^{-1}. \tag{2.17}
\]

Other formal Laplace-Borel transform pairs are given in Table 2.1. (See [24, Example 2.3.9] for additional discussion related to this example.)
Table 2.1. Some formal Laplace-Borel transform pairs.

<table>
<thead>
<tr>
<th>$F_c$</th>
<th>$\mathcal{L}_f[F_c]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_c : u \mapsto 1$</td>
<td>${F_c}$</td>
</tr>
<tr>
<td>$F_c : u \mapsto t^n$</td>
<td>$(n!x_0^n)$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \left( \sum_{i=0}^{n-1} \frac{(-1)^i}{i!} a_i t^i \right) e^{at}$</td>
<td>$(1 - ax_0)^{-n}$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \frac{1}{n!} \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)^n$</td>
<td>$(x_{i_1} + x_{i_2} + \cdots + x_{i_k})^n$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \sum_{n \geq 0} \frac{a_n}{n!} \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)^n$</td>
<td>$\sum_{n \geq 0} a_n (x_{i_1} + x_{i_2} + \cdots + x_{i_k})^n$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \exp \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)$</td>
<td>$(x_{i_1} + x_{i_2} + \cdots + x_{i_k})^*$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \exp \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)$</td>
<td>$\frac{x_{i_1} + x_{i_2} + \cdots + x_{i_k}}{[1 - (x_{i_1} + x_{i_2} + \cdots + x_{i_k})]^2}$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \cos \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)$</td>
<td>$\frac{1}{1 + (x_{i_1} + x_{i_2} + \cdots + x_{i_k})^2}$</td>
</tr>
<tr>
<td>$F_c : u \mapsto \sin \left( \int_{t_0}^{t} \sum_{j=1}^{k} u_{ij}(\tau) d\tau \right)$</td>
<td>$\frac{x_{i_1} + x_{i_2} + \cdots + x_{i_k}}{1 + (x_{i_1} + x_{i_2} + \cdots + x_{i_k})^2}$</td>
</tr>
</tbody>
</table>

Example 2.9. Let $X = \{x_0, x_1, \ldots, x_m\}$. Suppose $F_c$ has the generating series $c = \sum_{\eta \in X^*} \eta$, and $F_\xi$ is given for some fixed word $\xi \in X^*$. Then

$$\mathcal{L}_f[F_c \cdot F_\xi] = \mathcal{L}_f[F_c] \uplus \mathcal{L}_f[F_\xi] = c \uplus \xi$$

where $\binom{\gamma}{\xi}$ denotes the number of subwords of $\gamma$ which are equal to $\xi$ (see [21, page 127]).

3. The formal Laplace-Borel transform and the composition product

In this section, the formal Laplace-Borel transform of the composition of two Fliess operators is considered. This leads to an important semigroup isomorphism between the set of all Fliess operators and the set of all locally convergent formal power series with compatible dimensions. In system theory applications, this analysis is useful for models consisting of cascaded nonlinear input-output systems. The definition of the composition product first appeared in [4, 5]. Its set of known properties was significantly expanded in [13–17, 20]. The definition is constructed recursively in terms of the shuffle product.
Definition 3.1. For any \( c \in \mathbb{R}^\ell \langle \langle X \rangle \rangle \) and \( d \in \mathbb{R}^m \langle \langle X \rangle \rangle \), the composition product is defined as

\[
c \circ d = \sum_{\eta \in X^*} (c, \eta) \eta \circ d,
\]

where

\[
\eta \circ d = \begin{cases} 
|\eta|_{x_i} = 0, & \forall i \neq 0, \\
\sum_{n=1}^{\eta|_{x_i}} d_{ij} \bigcup (\eta' \circ d) : \eta = x_0^n x_i \eta', & n \geq 0, i \neq 0.
\end{cases}
\]

(Here \( |\eta|_{x_i} \) denotes the number of symbols in \( \eta \) equivalent to \( x_i \) and \( d_i : \xi \mapsto (d, \xi)_i \), the \( i \)th component of \( (d, \xi) \).)

Consequently, if

\[
\eta = x_0^{n_0} x_i x_{i1}^{-1} \cdots x_0^{n_1} x_i x_{i2}^{-1} \cdots,
\]

where \( i_j \neq 0 \) for \( j = 1, \ldots, k \), it follows that

\[
\eta \circ d = x_0^{n_0+1} \bigcup \sum_{n=1}^{\eta|_{x_i}} d_{ij} \bigcup x_0^{n_1+1} \bigcup \cdots \bigcup x_0^{n_k+1} \bigcup \cdots.
\]

Alternatively, for any \( \eta \in X^* \) one can uniquely define a set of right factors \( \{ \eta_0, \eta_1, \ldots, \eta_k \} \) of \( \eta \) by the iteration

\[
\eta_{j+1} = x_0^{n_{j+1}} x_i x_{ij} \eta_j, \quad \eta_0 = x_0^{n_0}, \quad i_j \neq 0,
\]

so that \( \eta = \eta_k \) with \( k = |\eta| - |\eta|_{x_0} \). In which case, \( \eta \circ d = \eta_k \circ d \), where \( \eta_{j+1} \circ d = x_0^{n_{j+1}+1} \bigcup (\eta_j \circ d) \) and \( \eta_0 \circ d = x_0^{n_0} \). It was shown in [15, 16] that the composition product of two series is always well defined since the family of series \( \{ \eta \circ d : \eta \in X^* \} \) is locally finite for any fixed \( d \in \mathbb{R}^m \langle \langle X \rangle \rangle \).

It is easily verified that for any series \( c, d, e \in \mathbb{R}^m \langle \langle X \rangle \rangle \), the composition product is left distributive over addition. That is,

\[
(c + d) \circ e = c \circ e + d \circ e,
\]

but in general \( c \circ (d + e) \neq c \circ d + c \circ e \). An exception is the class of series called linear series. A series \( c \) is linear if

\[
supp(c) \subseteq \{ \eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0}, i \in \{1, 2, \ldots, m\}, n_1, n_0 \geq 0 \}.
\]

The composition product is associative, that is, \( (c \circ d) \circ e = c \circ (d \circ e) \); hence \( (\mathbb{R}^m \langle \langle X \rangle \rangle, \circ) \) forms a semigroup [4, 20]. In [16], it is shown that the composition product of two locally convergent formal power series is always locally convergent; therefore the set \( \mathbb{R}^m_{LC} \langle \langle X \rangle \rangle \) is closed under composition, and \( (\mathbb{R}^m_{LC} \langle \langle X \rangle \rangle, \circ) \) also forms a semigroup. The central property of the composition product is its relationship to the composition of two Fliess operators. Namely, for any \( c \in \mathbb{R}^\ell_{LC} \langle \langle X \rangle \rangle \) and \( d \in \mathbb{R}^m_{LC} \langle \langle X \rangle \rangle \), the cascade connection of two
Fliess operators is always another Fliess operator with generating series \( c \circ d \), that is,
\[
F_c \circ F_d = F_{cd}.
\] (3.8)

Therefore, \( \mathcal{F} \) forms a semigroup under operator composition when \( \ell = m \). The following theorem shows that composition is preserved in a natural sense under the formal Laplace-Borel transform.

**Theorem 3.2.** Let \( X = \{x_0, x_1, \ldots, x_m\} \). For any \( c \in \mathbb{R}_L^\ell \langle \langle X \rangle \rangle \) and \( d \in \mathbb{R}_L^m \langle \langle X \rangle \rangle \),
\[
\mathcal{L}_f(F_c \circ F_d) = \mathcal{L}_f(F_c) \circ \mathcal{L}_f(F_d),
\]
and
\[
\mathcal{B}_f(c \circ d) = \mathcal{B}_f(c) \circ \mathcal{B}_f(d).
\] (3.9)

**Proof.** The proof follows directly from the definitions. For any well-defined \( F_c \) and \( F_d \),
\[
\mathcal{L}_f(F_c \circ F_d) = \mathcal{L}_f(F_{cd}) = c \circ d = \mathcal{L}_f(F_c) \circ \mathcal{L}_f(F_d).
\] (3.10)

Similarly, for any locally convergent formal power series \( c \) and \( d \),
\[
\mathcal{B}_f(c \circ d) = F_{cd} = F_c \circ F_d = \mathcal{B}_f(c) \circ \mathcal{B}_f(d).
\] (3.11)

From Theorem 3.2, when \( \ell = m \), the formal Laplace-Borel transform provides an isomorphism between the two semigroups \( (\mathcal{F}, \circ) \) and \( (\mathbb{R}_L^m \langle \langle X \rangle \rangle, \circ) \), as shown in Figure 3.1.

**Example 3.3.** Let \( X = \{x_0, x_1, x_2\} \), \( F_c[u](t) = \cos(\int_0^t u_1(t) + u_2(t)dt) \), and \( d = (d_1 \ d_2)^T \in \mathbb{R}_L^2 \langle \langle X \rangle \rangle \). Define
\[
F_c[u](t) = (F_c \circ F_d)[u](t) = \cos \left( \int_0^t F_{d_1}[u](t) + F_{d_2}[u](t)dt \right).
\] (3.12)

The formal Laplace transform of \( F_c \) is then
\[
\mathcal{L}_f[F_c] = c \circ d = \sum_{i=0}^\infty (-1)^i (x_1 + x_2)^{2i} \circ d
\]
\[
= \left[ - (x_1 + x_2)^2 \right]^{*} \circ d = \frac{1}{1 + (x_1 + x_2)^2} \circ d.
\] (3.13)
Example 3.4. Let $X = \{x_0, x_1, \ldots, x_m\}$ and $c \in \mathbb{R}^m_{\mathcal{LC}}\langle \langle X \rangle \rangle$. It is easily verified by induction that for $n \geq 1$,

$$x_i^n \circ c = \frac{1}{n!} (x_0 c_1)^{\cdot^{\cdot^n}}, \quad i = 1, 2, \ldots, m,$$

where $(\cdot)^{\cdot^{\cdot^n}}$ denotes the shuffle power.

Applying the formal Borel transform to both sides of this identity gives

$$\mathcal{B}_f[x_i^n \circ c] = \mathcal{B}_f \left[ \frac{1}{n!} (x_0 c_1)^{\cdot^{\cdot^n}} \right]$$

$$= \frac{1}{n!} \left[ \mathcal{B}_f(x_0 c_1) \right]^n$$

$$= \frac{1}{n!} \left[ \int_0^t F_c[u](\tau) d\tau \right]^n. \quad (3.15)$$

Example 3.5. First consider the linear ordinary differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{n-1} b_i \frac{d^i u(t)}{dt^i} \quad (3.16)$$

with initial conditions $y^{(i)}(0) = 0$, $u^{(i)}(0) = 0$, and where $a_i, b_i \in \mathbb{R}$ for $i = 0, 1, \ldots, n - 1$. Integrate both sides of the equation $n$ times and assume there exists a $c \in \mathbb{R}^m_{\mathcal{LC}}\langle \langle X \rangle \rangle$ such that $y = F_c[u]$. Applying the formal Laplace transform to both sides of the equation gives

$$\left( \delta + \sum_{i=0}^{n-1} a_i x_0^{n-1-i} x_1 \right) \circ c = \sum_{i=0}^{n-1} b_i x_0^{n-1-i} x_1, \quad (3.17)$$

$$\left( 1 + \sum_{i=0}^{n-1} a_i x_0^{n-i} \right) c = \sum_{i=0}^{n-1} b_i x_0^{n-1-i} x_1.$$  

Therefore,

$$c = \left( 1 + \sum_{i=0}^{n-1} a_i x_0^{n-i} \right)^{-1} \sum_{i=0}^{n-1} b_i x_0^{n-1-i} x_1. \quad (3.18)$$

Rephrased in the language of the integral Laplace transform, this is equivalent to

$$Y(s) = \left( 1 + \sum_{i=0}^{n-1} a_i \frac{1}{s^{n-i}} \right)^{-1} \left( \sum_{i=0}^{n-1} b_i \frac{1}{s^{n-i}} \right) U(s)$$

$$= \left( s^n + \sum_{i=0}^{n-1} a_i s^i \right)^{-1} \left( \sum_{i=0}^{n-1} b_i s^i \right) U(s). \quad (3.19)$$

Now consider the nonlinear differential equation

$$\frac{d^n y(t)}{dt^n} + \sum_{i=0}^{n-1} a_i \frac{d^i y(t)}{dt^i} + \sum_{j=2}^k p_j u(t) y^j(t) = \sum_{i=0}^{n-1} b_i \frac{d^i u(t)}{dt^i} \quad (3.20)$$
with \( y^{(i)}(0) = 0, u^{(i)}(0) = 0 \), and where \( a_i, b_i, p_j \in \mathbb{R} \) for \( i = 0, 1, \ldots, n - 1 \) and \( j = 2, \ldots, k \).

Again integrate both sides of the equation \( n \) times and assume \( y = F_c[u] \). Applying the formal Laplace transform gives

\[
\left( 1 + \sum_{i=0}^{n-1} a_i x_0^{n-i} \right) c + \sum_{j=2}^{k} p_j x_0^{n-1} x_1 \left( c^{\nu}:j \right) = \sum_{i=0}^{n-1} b_i x_0^{n-1-i} x_1. \tag{3.21}
\]

As in [12], a recursive procedure can be applied to solve the algebraic equation iteratively so that

\[
c = c_1 + c_2 + \cdots \tag{3.22}
\]

with

\[
c_1 = \left( 1 + \sum_{i=0}^{n-1} a_i x_0^{n-i} \right)^{-1} \sum_{i=0}^{n-1} b_i x_0^{n-1-i} x_1 \tag{3.23}
\]

and for \( n \geq 2 \)

\[
c_n = \left( 1 + \sum_{i=0}^{n-1} a_i x_0^{n-i} \right)^{-1} x_0^{n-1} x_1 \sum_{j=2}^{k} p_j \sum_{\nu_1 \geq 1, \nu_2 \geq 1} \cdots \sum_{\nu_j = n} c_{\nu_1} \cdots c_{\nu_j}. \tag{3.24}
\]

4. Operational calculus for the output response of Fliess operators

In [7, 12, 19], an operational calculus was developed to compute the output response of an analytic nonlinear system represented by a Fliess operator. In this section, the formal Laplace-Borel transform pair in conjunction with the composition product are employed to produce a clear and compact interpretation of this methodology.

**Theorem 4.1.** Let \( F_c \in \mathcal{F} \) and let \( u \) be an analytic function with formal Laplace transform \( c_u \in \mathbb{R}^m_{IC}(\langle X_0 \rangle) \). Then the output function \( y = F_c[u] \) is analytic and has formal Laplace transform \( c_y = c \circ c_u \in \mathbb{R}^m_{IC}(\langle X_0 \rangle) \).

**Proof.** The analyticity of \( y \) follows from [24, Lemma 2.3.8]. Let \( c_y \) denote its formal Laplace transform. Local convergence is preserved under the composition product; therefore the formal power series \( c \circ c_u \) is locally convergent, and, for any admissible input \( v \in B^m_p(R)[t_0, t_0 + T], \)

\[
y = F_{c_y}[v] = F_c[F_{c_u}[v]] = F_{c \circ c_u}[v]. \tag{4.1}
\]

Applying the formal Laplace transform to both sides of the equation gives \( c_y = c \circ c_u \). □

The following examples further illustrate the applications of this result for computing the output response of an analytic input-output system.

**Example 4.2.** Consider the linear time-invariant system \( y(t) = \int_0^t h(t - \tau) u(\tau) d\tau \), where \( h \) is analytic at \( t = 0 \). Then \( y = F_c[u] \) with \( (c, x_0^k x_1) = h^{(k)}(0) \), \( k \geq 0 \), and zero otherwise.
If \( u(t) = \sum_{k \geq 0} (c_u, x_0^k) t^k / k! \), then it follows that \( y(t) = \sum_{n \geq 0} (c_y, x_0^n) t^n / n! \), where

\[
    c_y = c \circ c_u = \sum_{k \geq 0} (c, x_0^k x_1) x_0^k x_1 \circ c_u \quad \text{(4.2)}
\]

Therefore,

\[
    (c_y, x_0^n) = \sum_{k=0}^{n-1} (c, x_0^k x_1) (c_u, x_0^{n-1-k}) \quad n \geq 1, \quad \text{(4.3)}
\]

which is the conventional convolution sum.

**Example 4.3.** Consider the simple Wiener system shown in Figure 4.1 where \( z(0) = 0 \). The mapping \( u \mapsto y \) can be written as

\[
    y(t) = \sum_{n=0}^{\infty} \left( E_{x_1} [u](t) \right)^n = \sum_{n=0}^{\infty} F_{x_1^{\otimes n}} [u](t) = \sum_{n=0}^{\infty} n! E_{x_1^n} [u](t). \quad \text{(4.4)}
\]

Therefore \( y = F_x [u] \) with \( c = \sum_{n=0}^{\infty} n! x_1^n \). When \( u(t) = t^k / k! \), for example, the formal Laplace transform of \( u \) is \( c_u = x_1^k \). From Theorem 4.1 and (3.14) it follows that

\[
    c_y = \sum_{n=0}^{\infty} n! x_1^n \circ x_0^k = \sum_{n=0}^{\infty} (x_0^{k+1})^{\otimes n} = \sum_{n=0}^{\infty} \frac{(k+1)n!}{(k+1)!} x_0^{(k+1)n}. \quad \text{(4.5)}
\]

Consequently, the output response is

\[
    y(t) = \sum_{n=0}^{\infty} \frac{(k+1)n!}{((k+1)n)!} \frac{t^{(k+1)n}}{((k+1)n)!} = \sum_{n=0}^{\infty} \frac{t^{(k+1)n}}{(k+1)!} = \frac{1}{1 - t^{k+1}/(k+1)!}. \quad \text{(4.6)}
\]

**5. Conclusion**

In this paper, the formal Laplace-Borel transform of a Fliess operator is defined and developed. This concept provides an isomorphism between the semigroup of all Fliess operators under composition and the semigroup of all locally convergent formal power series under the composition product. As an application in system theory, an explicit relationship is derived between the formal Laplace-Borel transforms of the input and output functions of a Fliess operator, which is a compact interpretation of the operational calculus of Fliess.
References


14 The formal Laplace-Borel transform of Fliess operators


Yaqin Li: Department of Electrical and Computer Engineering, University of Memphis, Memphis, TN 38152, USA  
*E-mail address*: yaqinli@memphis.edu

W. Steven Gray: Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, VA 23529, USA  
*E-mail address*: gray@ece.odu.edu
Special Issue on
Boundary Value Problems on Time Scales

Call for Papers

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

In recent years, the study of dynamic equations has led to several important applications, for example, in the study of insect population models, neural network, heat transfer, and epidemic models. This special issue will contain new researches and survey articles on Boundary Value Problems on Time Scales. In particular, it will focus on the following topics:

- Existence, uniqueness, and multiplicity of solutions
- Comparison principles
- Variational methods
- Mathematical models
- Biological and medical applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/ade/guidelines.html. Authors should follow the Advances in Difference Equations manuscript format described at the journal site http://www.hindawi.com/journals/ade/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>April 1, 2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>July 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>October 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Alberto Cabada, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; alberto.cabada@usc.es

Guest Editor

Victoria Otero-Espinor, Departamento de Análise Matemática, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain; mvictoria.otero@usc.es

Hindawi Publishing Corporation
http://www.hindawi.com