EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR DELAYED PREDATOR-PREY PATCH SYSTEMS WITH STOCKING

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A sufficient condition is derived for the existence of positive periodic solutions for a delayed predator-prey patch system with stocking. Some known results are improved.

1. Introduction

Predator-prey systems have been studied extensively. See, for instance, [1, 6, 8–10] and the references cited therein. Most of the previous papers focused on the predator-prey systems without stocking. Brauer and Soudack [2, 3] studied some predator-prey systems under constant rate stocking. To our knowledge, few papers have been published on the existence of positive periodic solutions for delayed predator-prey patch systems with periodic stocking.

In this paper, we investigate the following predator-prey system with stocking:

$$\begin{aligned} x_1'(t) &= x_1(t)\big(a_1(t) - b_1(t)x_1(t) - c(t)y(t)\big) + D_1(t)\big(x_2(t-\tau_1(t)) - x_1(t)\big) + S_1(t), \\ x_2'(t) &= x_2(t)\big(a_2(t) - b_2(t)x_2(t)\big) + D_2(t)\big(x_1(t-\tau_2(t)) - x_2(t)\big) + S_2(t), \\ y'(t) &= y(t)\left(-d(t) + p(t)x_1(t) - q(t)y(t) - \beta(t)\int_{-\tau}^{0} k(s)y(t+s)\,ds\right) + S_3(t), \end{aligned} \tag{1.1}$$

with the initial conditions

$$\begin{aligned} x_1(s) &= \varphi_1(s) \geq 0, \quad s \in [-\sigma, 0],\ \varphi_1(0) > 0, \\ x_2(s) &= \varphi_2(s) \geq 0, \quad s \in [-\sigma, 0],\ \varphi_2(0) > 0, \\ y(s) &= \psi(s) \geq 0, \quad s \in [-\sigma, 0],\ \psi(0) > 0, \end{aligned} \tag{1.2}$$

where x_1 and y are the population densities of prey species x and predator species y in patch 1, and x_2 is the density of species x in patch 2. Predator species y is confined to

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patch 1, while the prey species x can diffuse between two patches. $D_i(t)$ $(i = 1,2)$ are diffusion coefficients of species x. $S_i(t)$ $(i = 1,2,3)$ denote the stocking rates. $\varphi_1(s)$, $\varphi_2(s)$, and $\psi(s)$ are continuous on $[-\sigma,0]$, $\sigma = \max\{\tau, \sup_{t\in\mathbb{R}}\tau_1(t), \sup_{t\in\mathbb{R}}\tau_2(t)\}$. The delay $\tau_1(\tau_2)$ represents the time that species x migrates from patch 2 to patch 1 (patch 1 to patch 2).

When $S_i(t) \equiv 0$ $(i = 1,2,3)$, $\tau_i \equiv 0$ $(i = 1,2)$, system (1.1) was considered by Zhang and Wang [15], Song and Chen [11], and Chen et al. [5].

The purpose of this paper is to derive a set of easily verifiable conditions for the existence of positive periodic solutions of system (1.1). The method in this paper is different from those of [4, 12–14].

2. Existence of positive periodic solutions

To show the existence of solutions to the considered problems, we will use an abstract theorem developed [7]. We first state this abstract theorem.

For a fixed $\sigma \geq 0$, let $\mathbb{C} := \mathbb{C}([-\sigma,0];\mathbb{R}^n)$. If $x \in \mathbb{C}([\gamma-\sigma,\gamma+\delta];\mathbb{R}^n)$ for some $\delta > 0$ and $\gamma \in \mathbb{R}$, then $x_t \in \mathbb{C}$ for $t \in [\gamma,\gamma+\delta]$ is defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\sigma,0]$. The supremum norm in $\mathbb{C}$ is denoted by $\|\cdot\|_c$, that is, $\|\phi\|_c = \max_{\theta\in[-\sigma,0]}\|\phi(\theta)\|$ for $\phi \in \mathbb{C}$, where $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$, and $\|u\| = \sum_{i=1}^n |u_i|$ for $u = (u_1,\dots,u_n) \in \mathbb{R}^n$.

We consider the following functional differential equation:

$$\frac{dx(t)}{dt} = f(t,x_t), \tag{2.1}$$

where $f : \mathbb{R}\times\mathbb{C} \to \mathbb{R}^n$ is completely continuous, and there exists $T > 0$ such that for every $(t,\varphi) \in \mathbb{R}\times\mathbb{C}$, we have $f(t+T,\varphi) = f(t,\varphi)$.

The following lemma is a simple consequence of [7, Theorem 4.7.1].

Lemma 2.1. *Suppose that there exists a constant $M > 0$ such that*

(i) *for any $\lambda \in (0,1)$ and any T-periodic solution x of the system*

$$\frac{dx(t)}{dt} = \lambda f(t,x_t), \tag{2.2}$$

$\|x(t)\| < M$ *for* $t \in \mathbb{R}$;

(ii) $g(u) := (1/T)\int_0^T f(s,\hat{u})ds \neq 0$ *for $u \in \partial B_M(\mathbb{R}^n)$, where $B_M(\mathbb{R}^n) = \{u \in \mathbb{R}^n : \|u\| < M\}$, and $\hat{u}$ denotes the constant mapping from $[-\sigma,0]$ to $\mathbb{R}^n$ with the value $u \in \mathbb{R}^n$;*

(iii) *Brouwer degree* $\deg(g,B_M(\mathbb{R}^n)) \neq 0$.

Then there exists at least one T-periodic solution of the system

$$\frac{dx(t)}{dt} = f(t,x_t) \tag{2.3}$$

that satisfies $\sup_{t\in\mathbb{R}}\|x(t)\| < M$.

In the following, we set

$$\bar{g} = \frac{1}{T}\int_0^T g(t)dt, \qquad g^l = \min_{t\in[0,T]}|g(t)|, \qquad g^u = \max_{t\in[0,T]}|g(t)|, \tag{2.4}$$

where g is a continuous T-periodic function.

In system (1.1), we always assume the following.

(H_1) $a_i(t)$, $b_i(t)$, $D_i(t)$ $(i=1,2)$, $c(t)$, $d(t)$, $p(t)$, $q(t)$, and $\beta(t)$ are positive continuous T-periodic functions. $S_i(t)$ $(i=1,2,3)$, $\tau_i(t)$ $(i=1,2)$ are nonnegative continuous T-periodic functions. $\tau_i'(t)<1$ $(i=1,2)$, $t\in\mathbb{R}$.

(H_2) $k(s)\geq 0$ on $[-\tau,0]$ $(0\leq\tau<+\infty)$; and $k(s)$ is a piecewise continuous and normalized function such that $\int_{-\tau}^{0}k(s)ds=1$.

Set

$$K=\frac{\bar{q}+\bar{\beta}}{\bar{p}},$$

$$K^*=\left(\frac{a_1M_0-D_1M_0+S_1}{b_1M_0}\right)^l,\qquad K_i^*=\left(\frac{a_iM_0+S_i}{b_iM_0}\right)^l,\quad i=1,2,$$

$$M_0=\max\left\{\left(\frac{a_1+\sqrt{a_1^2+4b_1S_1}}{2b_1}\right)^u,\left(\frac{a_2+\sqrt{a_2^2+4b_2S_2}}{2b_2}\right)^u\right\},$$

$$m_0=\min\left\{\frac{(a_1/c)^l-\sqrt{(S_3/q)^u}}{b_1^u/c^l+(p/q)^u}\exp\left[-2T(\bar{D}_1+\bar{b}_1M_0+\bar{c}\widetilde{M}_0)\right],\left(\frac{a_2+\sqrt{a_2^2+4b_2S_2}}{2b_2}\right)^l\right\},$$

$$\widetilde{M}_0=\left(\frac{pM_0+\sqrt{p^2M_0^2+4qS_3}}{2q}\right)^u,$$

$$\widetilde{m}_0=\min\left\{\frac{K_1^*-\bar{d}/\bar{p}}{K+(c/b_1)^u},\frac{K_2^*-\bar{d}/\bar{p}}{K},\frac{K^*-\bar{d}/\bar{p}}{K+(c/b_1)^u}\right\}\exp\left[-2T(\bar{d}+\bar{q}\widetilde{M}_0+\bar{\beta}\widetilde{M}_0)\right].\tag{2.5}$$

Theorem 2.2. *In addition to* (H_1), (H_2), *assume further that system (1.1) satisfies one of the following assumptions:*

(H_3) $(a_1/c)^l>\sqrt{(S_3/q)^u}$, $K_i^*>\bar{d}/\bar{p}$ $(i=1,2)$;

(H_4) $(a_1/c)^l>\sqrt{(S_3/q)^u}$, $K^*>\bar{d}/\bar{p}$.

Then system (1.1) has at least one positive T-periodic solution, say $(x_1^*(t),x_2^*(t),y^*(t))^T$ such that

$$m_0\leq x_i^*(t)\leq M_0\quad(i=1,2),\qquad \widetilde{m}_0\leq y^*(t)\leq\widetilde{M}_0,\quad t\geq 0.\tag{2.6}$$

Proof. Consider the following system:

$$\begin{aligned}
u_1'(t)&=a_1(t)-D_1(t)-b_1(t)e^{u_1(t)}-c(t)e^{u_3(t)}+D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}+\frac{S_1(t)}{e^{u_1(t)}},\\
u_2'(t)&=a_2(t)-D_2(t)-b_2(t)e^{u_2(t)}+D_2(t)e^{u_1(t-\tau_2(t))-u_2(t)}+\frac{S_2(t)}{e^{u_2(t)}},\\
u_3'(t)&=-d(t)+p(t)e^{u_1(t)}-q(t)e^{u_3(t)}-\beta(t)\int_{-\tau}^{0}k(s)e^{u_3(t+s)}ds+\frac{S_3(t)}{e^{u_3(t)}},
\end{aligned}\tag{2.7}$$

where $a_i(t)$, $b_i(t)$, $D_i(t)$ $(i=1,2)$, $S_i(t)$ $(i=1,2,3)$, $c(t)$, $d(t)$, $p(t)$, $q(t)$, and $\beta(t)$ are the same as those in assumption (H_1), and τ, τ_i $(i=1,2)$ and $k(s)$ are the same as those in assumption (H_2). We first show that system (2.7) has one T-periodic solution.

Let $\mathbb{C}:=\mathbb{C}([-\sigma,0];\mathbb{R}^3)$. We define the following map:

$$
\begin{aligned}
&f:\mathbb{R}\times\mathbb{C}\longrightarrow\mathbb{R}^3,\quad f(t,\varphi)=\big(f_1(t,\varphi),f_2(t,\varphi),f_3(t,\varphi)\big),\quad \varphi=(\varphi_1,\varphi_2,\varphi_3)\in\mathbb{C},\\
&f_1(t,\varphi)=a_1(t)-D_1(t)-b_1(t)e^{\varphi_1(0)}-c(t)e^{\varphi_3(0)}+D_1(t)e^{\varphi_2(-\tau_1(t))-\varphi_1(0)}+\frac{S_1(t)}{e^{\varphi_1(0)}},\\
&f_2(t,\varphi)=a_2(t)-D_2(t)-b_2(t)e^{\varphi_2(0)}+D_2(t)e^{\varphi_1(-\tau_2(t))-\varphi_2(0)}+\frac{S_2(t)}{e^{\varphi_2(0)}},\\
&f_3(t,\varphi)=-d(t)+p(t)e^{\varphi_1(0)}-q(t)e^{\varphi_3(0)}-\beta(t)\int_{-\tau}^{0}k(s)e^{\varphi_3(s)}ds+\frac{S_3(t)}{e^{\varphi_3(0)}}.
\end{aligned}
\tag{2.8}
$$

Clearly, $f:\mathbb{R}\times\mathbb{C}\to\mathbb{R}^3$ is completely continuous. Now, the system (2.7) becomes

$$
\frac{du(t)}{dt}=f(t,u_t). \tag{2.9}
$$

Corresponding to

$$
\frac{du(t)}{dt}=\lambda f(t,u_t),\quad \lambda\in(0,1), \tag{2.10}
$$

we have

$$
\begin{aligned}
u_1'(t)&=\lambda\left[a_1(t)-D_1(t)-b_1(t)e^{u_1(t)}-c(t)e^{u_3(t)}+D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}+\frac{S_1(t)}{e^{u_1(t)}}\right],\\
u_2'(t)&=\lambda\left[a_2(t)-D_2(t)-b_2(t)e^{u_2(t)}+D_2(t)e^{u_1(t-\tau_2(t))-u_2(t)}+\frac{S_2(t)}{e^{u_2(t)}}\right],\\
u_3'(t)&=\lambda\left[-d(t)+p(t)e^{u_1(t)}-q(t)e^{u_3(t)}-\beta(t)\int_{-\tau}^{0}k(s)e^{u_3(t+s)}ds+\frac{S_3(t)}{e^{u_3(t)}}\right].
\end{aligned}
\tag{2.11}
$$

Suppose that $(u_1(t),u_2(t),u_3(t))^T$ is a T-periodic solution of system (2.11) for some $\lambda\in(0,1)$. Choose $t_i^M,t_i^m\in[0,T]$, $i=1,2,3$, such that

$$
u_i(t_i^M)=\max_{t\in[0,T]}u_i(t),\quad u_i(t_i^m)=\min_{t\in[0,T]}u_i(t),\quad i=1,2,3. \tag{2.12}
$$

Then, it is clear that

$$
u_i'(t_i^M)=0,\quad u_i'(t_i^m)=0,\quad i=1,2,3. \tag{2.13}
$$

From this and system (2.11), we obtain that

$$a_1(t_1^M)-D_1(t_1^M)-b_1(t_1^M)e^{u_1(t_1^M)}-c(t_1^M)e^{u_3(t_1^M)}+D_1(t_1^M)e^{u_2(t_1^M-\tau_1(t_1^M))-u_1(t_1^M)}+\frac{S_1(t_1^M)}{e^{u_1(t_1^M)}}=0, \tag{2.14}$$

$$a_2(t_2^M)-D_2(t_2^M)-b_2(t_2^M)e^{u_2(t_2^M)}+D_2(t_2^M)e^{u_1(t_2^M-\tau_2(t_2^M))-u_2(t_2^M)}+\frac{S_2(t_2^M)}{e^{u_2(t_2^M)}}=0, \tag{2.15}$$

$$-d(t_3^M)+p(t_3^M)e^{u_1(t_3^M)}-q(t_3^M)e^{u_3(t_3^M)}-\beta(t_3^M)\int_{-\tau}^{0}k(s)e^{u_3(t_3^M+s)}\,ds+\frac{S_3(t_3^M)}{e^{u_3(t_3^M)}}=0, \tag{2.16}$$

$$a_1(t_1^m)-D_1(t_1^m)-b_1(t_1^m)e^{u_1(t_1^m)}-c(t_1^m)e^{u_3(t_1^m)}+D_1(t_1^m)e^{u_2(t_1^m-\tau_1(t_1^m))-u_1(t_1^m)}+\frac{S_1(t_1^m)}{e^{u_1(t_1^m)}}=0, \tag{2.17}$$

$$a_2(t_2^m)-D_2(t_2^m)-b_2(t_2^m)e^{u_2(t_2^m)}+D_2(t_2^m)e^{u_1(t_2^m-\tau_2(t_2^m))-u_2(t_2^m)}+\frac{S_2(t_2^m)}{e^{u_2(t_2^m)}}=0. \tag{2.18}$$

Next we make the following claims.

Claim 1. For $u_i(t_i^M)$ $(i=1,2)$, one of the following cases holds:

$$u_2(t_2^M)\le u_1(t_1^M)\le M_1^*\le M_1, \tag{2.19}$$

$$u_1(t_1^M)< u_2(t_2^M)\le M_2^*\le M_1, \tag{2.20}$$

where $M_1:=\max\{M_1^*,M_2^*\}$, $M_j^*:=\ln((a_j+\sqrt{a_j^2+4b_jS_j})/2b_j)^u$, $j=1,2$.

There are two cases to consider.

Case 1. Assume that $u_1(t_1^M)\ge u_2(t_2^M)$; then $u_1(t_1^M)\ge u_2(t_1^M-\tau_1(t_1^M))$.

From this and (2.14), we have

$$b_1(t_1^M)e^{u_1(t_1^M)}\le a_1(t_1^M)+\frac{S_1(t_1^M)}{e^{u_1(t_1^M)}}. \tag{2.21}$$

That is,

$$b_1(t_1^M)e^{2u_1(t_1^M)}-a_1(t_1^M)e^{u_1(t_1^M)}-S_1(t_1^M)\le 0. \tag{2.22}$$

Therefore,

$$e^{u_1(t_1^M)}\le\frac{a_1(t_1^M)+\sqrt{a_1^2(t_1^M)+4b_1(t_1^M)S_1(t_1^M)}}{2b_1(t_1^M)}\le\left(\frac{a_1+\sqrt{a_1^2+4b_1S_1}}{2b_1}\right)^u. \tag{2.23}$$

Hence,

$$u_2(t_2^M)\le u_1(t_1^M)\le\ln\left(\frac{a_1+\sqrt{a_1^2+4b_1S_1}}{2b_1}\right)^u. \tag{2.24}$$

Case 2. Assume that $u_1(t_1^M) < u_2(t_2^M)$; then $u_1(t_2^M - \tau_2(t_2^M)) < u_2(t_2^M)$.
From this and (2.15), we have

$$b_2(t_2^M)e^{u_2(t_2^M)} \leq a_2(t_2^M) + \frac{S_2(t_2^M)}{e^{u_2(t_2^M)}}. \tag{2.25}$$

By a similar argument to Case 1, we have

$$u_1(t_1^M) < u_2(t_2^M) \leq \ln\left(\frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2}\right)^u. \tag{2.26}$$

It follows from (2.24) and (2.26) that Claim 1 holds.

Claim 2.

$$u_3(t_3^M) \leq \ln\left(\frac{pM_0 + \sqrt{p^2M_0^2 + 4qS_3}}{2q}\right)^u := M_2, \tag{2.27}$$

where $M_0 = e^{M_1}$.
By (2.16), we have

$$q(t_3^M)e^{u_3(t_3^M)} \leq p(t_3^M)e^{u_1(t_3^M)} + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}} \leq p(t_3^M)e^{u_1(t_1^M)} + \frac{S_3(t_3^M)}{e^{u_3(t_3^M)}}. \tag{2.28}$$

That is,

$$q(t_3^M)e^{2u_3(t_3^M)} - p(t_3^M)e^{u_1(t_1^M)}e^{u_3(t_3^M)} - S_3(t_3^M) \leq 0. \tag{2.29}$$

Therefore,

$$e^{u_3(t_3^M)} \leq \frac{p(t_3^M)e^{u_1(t_1^M)} + \sqrt{p^2(t_3^M)e^{2u_1(t_1^M)} + 4q(t_3^M)S_3(t_3^M)}}{2q(t_3^M)}, \tag{2.30}$$

which implies that Claim 2 holds.
Claim 3. For $u_i(t_i^m)(i = 1,2)$, one of the following cases holds:

$$\begin{gathered} m_1 \leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m), \\ m_1 \leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m), \end{gathered} \tag{2.31}$$

where

$$\begin{gathered} m_1 := \min\{m_1^* - 2T(\bar{D}_1 + \bar{b}_1M_0 + \bar{c}\widetilde{M}_0), m_2^*\}, \\ m_1^* := \ln\frac{(a_1/c)^l - \sqrt{(S_3/q)}^u}{b_1^u/c^l + (p/q)^u}, \\ m_2^* := \ln\left(\frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2}\right)^l. \end{gathered} \tag{2.32}$$

There are two cases to consider.

Case 1. Assume that $u_1(t_1^m) \le u_2(t_2^m)$; then $u_1(t_1^m) \le u_2(t_1^m - \tau_1(t_1^m))$.

From this and (2.17), we have

$$a_1(t_1^m) \le b_1(t_1^m)e^{u_1(t_1^m)} + c(t_1^m)e^{u_3(t_1^m)} \le b_1(t_1^m)e^{u_1(t_1^M)} + c(t_1^m)e^{u_3(t_3^M)}. \tag{2.33}$$

From (2.30), by using the inequality

$$(a+b)^{1/2} < a^{1/2} + b^{1/2}, \quad a > 0,\ b > 0, \tag{2.34}$$

we have

$$e^{u_3(t_3^M)} < \frac{p(t_3^M)e^{u_1(t_1^M)} + \sqrt{q(t_3^M)S_3(t_3^M)}}{q(t_3^M)}. \tag{2.35}$$

From this and (2.33), we have

$$a_1(t_1^m) \le \left[b_1(t_1^m) + \frac{c(t_1^m)p(t_3^M)}{q(t_3^M)}\right]e^{u_1(t_1^M)} + c(t_1^m)\sqrt{\frac{S_3(t_3^M)}{q(t_3^M)}}, \tag{2.36}$$

which implies

$$\left(\frac{a_1}{c}\right)^l \le \left[\frac{b_1^u}{c^l} + \left(\frac{p}{q}\right)^u\right]e^{u_1(t_1^M)} + \sqrt{\left(\frac{S_3}{q}\right)^u}. \tag{2.37}$$

That is,

$$u_1(t_1^M) \ge \ln\frac{(a_1/c)^l - \sqrt{(S_3/q)^u}}{b_1^u/c^l + (p/q)^u} := m_1^*. \tag{2.38}$$

From the first equation of system (2.11), we obtain that

$$\begin{aligned}
&\int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_1(t)}}dt \\
&\quad = \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt, \\
&\int_0^T |u_1'(t)|\,dt < \int_0^T a_1(t)dt + \int_0^T D_1(t)e^{u_2(t-\tau_1(t))-u_1(t)}dt + \int_0^T \frac{S_1(t)}{e^{u_1(t)}}dt \\
&\quad + \int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt.
\end{aligned} \tag{2.39}$$

It follows that

$$\begin{aligned}\int_0^T |u_1'(t)|\,dt &< 2\left[\int_0^T D_1(t)dt + \int_0^T b_1(t)e^{u_1(t)}dt + \int_0^T c(t)e^{u_3(t)}dt\right]\\ &\le 2\left[\int_0^T D_1(t)dt + e^{M_1}\int_0^T b_1(t)dt + e^{M_2}\int_0^T c(t)dt\right]\\ &= 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0).\end{aligned} \tag{2.40}$$

From (2.38) and (2.40), we have

$$u_1(t_1^m) \ge u_1(t_1^M) - \int_0^T |u_1'(t)|\,dt \ge m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0). \tag{2.41}$$

Case 2. Assume that $u_1(t_1^m) > u_2(t_2^m)$; then $u_1(t_2^m - \tau_2(t_2^m)) > u_2(t_2^m)$.
From this and (2.18), we have

$$b_2(t_2^m)e^{u_2(t_2^m)} \ge a_2(t_2^m) + \frac{S_2(t_2^m)}{e^{u_2(t_2^m)}}, \tag{2.42}$$

which implies

$$e^{u_2(t_2^m)} \ge \frac{a_2(t_2^m) + \sqrt{a_2^2(t_2^m) + 4b_2(t_2^m)S_2(t_2^m)}}{2b_2(t_2^m)}. \tag{2.43}$$

That is,

$$u_2(t_2^m) \ge \ln\left(\frac{a_2 + \sqrt{a_2^2 + 4b_2S_2}}{2b_2}\right)^l := m_2^*. \tag{2.44}$$

It follows from (2.41) and (2.44) that Claim 3 holds.
Claim 4.

$$u_3(t_3^m) \ge \min\{m_3^*, m_4^*, m_5^*\} - 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0) := m_2, \tag{2.45}$$

where

$$m_3^* = \ln\frac{K_1^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}, \qquad m_4^* = \ln\frac{K_2^* - \bar{d}/\bar{p}}{K}, \qquad m_5^* = \ln\frac{K^* - \bar{d}/\bar{p}}{K + (c/b_1)^u}. \tag{2.46}$$

From the third equation of (2.11), we obtain

$$\begin{aligned}&\int_0^T p(t)e^{u_1(t)}dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}}dt = \int_0^T d(t)dt + \int_0^T q(t)e^{u_3(t)}dt + \int_0^T \beta(t)\int_{-\tau}^0 k(s)e^{u_3(t+s)}ds\,dt,\\ &\int_0^T |u_3'(t)|\,dt < \int_0^T p(t)e^{u_1(t)}dt + \int_0^T \frac{S_3(t)}{e^{u_3(t)}}dt + \int_0^T d(t)dt\\ &\qquad + \int_0^T q(t)e^{u_3(t)}dt + \int_0^T \beta(t)\int_{-\tau}^0 k(s)e^{u_3(t+s)}ds\,dt.\end{aligned} \tag{2.47}$$

It follows that

$$\begin{aligned}\int_0^T |u_3'(t)|\,dt &< 2\left[\int_0^T d(t)dt+\int_0^T q(t)e^{u_3(t)}dt+\int_0^T \beta(t)\int_{-\tau}^0 k(s)e^{u_3(t+s)}ds\,dt\right]\\ &\le 2\left[\int_0^T d(t)dt+e^{M_2}\int_0^T q(t)dt+e^{M_2}\int_0^T \beta(t)dt\right]\\ &= 2T(\bar{d}+\bar{q}\widetilde{M}_0+\bar{\beta}\widetilde{M}_0),\end{aligned}\tag{2.48}$$

$$[\bar{q}+\bar{\beta}]e^{u_3(t_3^M)} \ge \bar{p}e^{u_1(t_1^m)}-\bar{d}.\tag{2.49}$$

There are two cases to consider.

Case 1. Assume that the assumption (H_3) holds.

If $u_1(t_1^m) \le u_2(t_2^m)$, by (2.17), we have

$$\begin{aligned}e^{u_1(t_1^m)} &\ge \frac{a_1(t_1^m)-c(t_1^m)e^{u_3(t_1^m)}}{b_1(t_1^m)}+\frac{S_1(t_1^m)}{b_1(t_1^m)e^{u_1(t_1^m)}}\\ &\ge \frac{a_1(t_1^m)-c(t_1^m)e^{u_3(t_3^M)}}{b_1(t_1^m)}+\frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}.\end{aligned}\tag{2.50}$$

Substituting this into (2.49) gives

$$[\bar{q}+\bar{\beta}]e^{u_3(t_3^M)} \ge \frac{\bar{p}a_1(t_1^m)}{b_1(t_1^m)}-\frac{\bar{p}c(t_1^m)e^{u_3(t_3^M)}}{b_1(t_1^m)}+\frac{\bar{p}S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}-\bar{d},\tag{2.51}$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}}+\frac{\bar{\beta}}{\bar{p}}+\frac{c(t_1^m)}{b_1(t_1^m)}\right]e^{u_3(t_3^M)} \ge \frac{a_1(t_1^m)}{b_1(t_1^m)}+\frac{S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}-\frac{\bar{d}}{\bar{p}}.\tag{2.52}$$

Therefore,

$$\left[K+\left(\frac{c}{b_1}\right)^u\right]e^{u_3(t_3^M)} \ge K_1^*-\frac{\bar{d}}{\bar{p}}.\tag{2.53}$$

That is,

$$u_3(t_3^M) \ge \ln\frac{K_1^*-\bar{d}/\bar{p}}{K+(c/b_1)^u} := m_3^*.\tag{2.54}$$

It follows from (2.48) and (2.54)that

$$u_3(t_3^m) \ge u_3(t_3^M)-\int_0^T |u_3'(t)|\,dt \ge m_3^*-2T(\bar{d}+\bar{q}\widetilde{M}_0+\bar{\beta}\widetilde{M}_0).\tag{2.55}$$

If $u_1(t_1^m) > u_2(t_2^m)$, by (2.42), (2.49), and (2.19), we have

$$[\bar{q}+\bar{\beta}]e^{u_3(t_3^M)} \ge \bar{p}e^{u_2(t_2^m)}-\bar{d} \ge \frac{\bar{p}[a_2(t_2^m)+S_2(t_2^m)e^{-M_1}]}{b_2(t_2^m)}-\bar{d},\tag{2.56}$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}}+\frac{\bar{\beta}}{\bar{p}}\right]e^{u_3(t_3^M)} \geq \frac{a_2(t_2^m)+S_2(t_2^m)e^{-M_1}}{b_2(t_2^m)}-\frac{\bar{d}}{\bar{p}}. \tag{2.57}$$

Therefore,

$$Ke^{u_3(t_3^M)} \geq K_2^* - \frac{\bar{d}}{\bar{p}}. \tag{2.58}$$

That is,

$$u_3(t_3^M) \geq \ln\frac{K_2^*-\bar{d}/\bar{p}}{K} := m_4^*. \tag{2.59}$$

From (2.48) and (2.59), we have

$$u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u_3'(t)|\,dt \geq m_4^* - 2T(\bar{d}+\bar{q}\widetilde{M}_0+\bar{\beta}\widetilde{M}_0). \tag{2.60}$$

Case 2. Assume that the assumption (H_4) holds.

From (2.17), we have

$$\begin{aligned} b_1(t_1^m)e^{u_1(t_1^m)} &\geq a_1(t_1^m)-D_1(t_1^m)-c(t_1^m)e^{u_3(t_1^m)}+\frac{S_1(t_1^m)}{e^{u_1(t_1^m)}} \\ &\geq a_1(t_1^m)-D_1(t_1^m)-c(t_1^m)e^{u_3(t_3^M)}+\frac{S_1(t_1^m)}{e^{M_1}}. \end{aligned} \tag{2.61}$$

Therefore,

$$e^{u_1(t_1^m)} \geq \frac{a_1(t_1^m)-D_1(t_1^m)-c(t_1^m)e^{u_3(t_3^M)}+S_1(t_1^m)e^{-M_1}}{b_1(t_1^m)}. \tag{2.62}$$

Substituting this into (2.49) gives

$$[\bar{q}+\bar{\beta}]e^{u_3(t_3^M)} \geq \frac{\bar{p}[a_1(t_1^m)-D_1(t_1^m)]}{b_1(t_1^m)}-\frac{\bar{p}c(t_1^m)e^{u_3(t_3^M)}}{b_1(t_1^m)}+\frac{\bar{p}S_1(t_1^m)}{b_1(t_1^m)e^{M_1}}-\bar{d}, \tag{2.63}$$

which implies

$$\left[\frac{\bar{q}}{\bar{p}}+\frac{\bar{\beta}}{\bar{p}}+\frac{c(t_1^m)}{b_1(t_1^m)}\right]e^{u_3(t_3^M)} \geq \frac{a_1(t_1^m)-D_1(t_1^m)+S_1(t_1^m)e^{-M_1}}{b_1(t_1^m)}-\frac{\bar{d}}{\bar{p}}. \tag{2.64}$$

Therefore,

$$\left[K+\left(\frac{c}{b_1}\right)^u\right]e^{u_3(t_3^M)} \geq K^* - \frac{\bar{d}}{\bar{p}}. \tag{2.65}$$

That is,

$$u_3(t_3^M) \geq \ln\frac{K^*-\bar{d}/\bar{p}}{K+(c/b_1)^u} := m_5^*. \tag{2.66}$$

It follows from(2.48) and (2.66)that

$$u_3(t_3^m) \geq u_3(t_3^M) - \int_0^T |u_3'(t)|\,dt \geq m_5^* - 2T(\bar{d} + \bar{q}\widetilde{M}_0 + \bar{\beta}\widetilde{M}_0). \tag{2.67}$$

It follows from (2.55), (2.60), and (2.67) that Claim 4 holds.

Clearly, one of the following inequalities holds:

(i) $M_1^* > m_2^*$,

(ii) $M_1^* \leq m_2^*$.

Since $m_1^* < M_1^*$ and $m_2^* \leq M_2^*$, (ii) implies $M_2^* > m_1^*$. Thus, according to Claims 1–3, one of the following four cases must hold:

(P_1) $m_1 \leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m)$, $u_2(t_2^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1$;

(P_2) $m_1 \leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m)$, $u_2(t_2^M) \leq u_1(t_1^M) \leq M_1^* \leq M_1$;

(P_3) $m_1 \leq m_1^* - 2T(\bar{D}_1 + \bar{b}_1 M_0 + \bar{c}\widetilde{M}_0) \leq u_1(t_1^m) \leq u_2(t_2^m)$, $u_1(t_1^M) < u_2(t_2^M) \leq M_2^* \leq M_1$;

(P_4) $m_1 \leq m_2^* \leq u_2(t_2^m) < u_1(t_1^m)$, $u_1(t_1^M) < u_2(t_2^M) \leq M_2^* \leq M_1$.

From this and Claims 3 and 4, we have

$$\max_{t\in[0,T]} |u_i(t)| \leq \max\{|M_1|, |M_2|, |m_1|, |m_2|\} := M^*, \quad i = 1,2,3. \tag{2.68}$$

Obviously, M^* is independent of λ.

Set

$$B_i^* := \bar{a}_i + \sqrt{(\bar{a}_i)^2 + 4\bar{b}_i\bar{S}_i}, \quad i = 1,2. \tag{2.69}$$

Take sufficiently large M such that

$$\begin{gathered} M > 3\max\{M^*, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|\}, \\ M > |v_1^*| + |v_2^*| + |v_3^*|, \end{gathered} \tag{2.70}$$

where

$$\begin{gathered} v_1^* = \ln\frac{B_1^*}{2\bar{b}_1}, \qquad v_2^* = \ln\frac{B_2^*}{2\bar{b}_2}, \\ v_3^* = \ln\frac{\bar{p}B_1^* + \sqrt{[\bar{p}B_1^*]^2 + 16(\bar{b}_1)^2[\bar{q}+\bar{\beta}]\bar{S}_3}}{4\bar{b}_1[\bar{q}+\bar{\beta}]}. \end{gathered} \tag{2.71}$$

Clearly, the condition (i) in Lemma 2.1 is satisfied by system (2.7).

Define $H(u_1,u_2,u_3,\mu) : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ by

$$H(u_1,u_2,u_3,\mu) = \begin{pmatrix} \bar{a}_1 - \bar{b}_1 e^{u_1} + \dfrac{\bar{S}_1}{e^{u_1}} \\ \bar{a}_2 - \bar{b}_2 e^{u_2} + \dfrac{\bar{S}_2}{e^{u_2}} \\ \bar{p}e^{u_1} - [\bar{q}+\bar{\beta}]e^{u_3} + \dfrac{\bar{S}_3}{e^{u_3}} \end{pmatrix} + \mu\begin{pmatrix} \bar{D}_1 e^{u_2-u_1} - \bar{c}e^{u_3} - \bar{D}_1 \\ \bar{D}_2 e^{u_1-u_2} - \bar{D}_2 \\ -\bar{d} \end{pmatrix}. \tag{2.72}$$

We show that

$$H(u_1,u_2,u_3,\mu) \neq 0 \quad \text{for any } u = (u_1,u_2,u_3) \in \partial B_M(\mathbb{R}^3),\ \mu \in [0,1]. \tag{2.73}$$

Indeed, assume to the contrary, that

$$H(u_1^*,u_2^*,u_3^*,\mu^*) = 0 \quad \text{for some } u^* = (u_1^*,u_2^*,u_3^*) \in \partial B_M(\mathbb{R}^3),\ \mu^* \in [0,1]. \tag{2.74}$$

Then, there exist $t_i \in [0,T]$, $i = 1,2$, such that

$$\begin{gathered} a_1(t_1) - b_1(t_1)e^{u_1^*} + \frac{S_1(t_1)}{e^{u_1^*}} + \mu^* D_1(t_1)e^{u_2^*-u_1^*} - \mu^* c(t_1)e^{u_3^*} - \mu^* D_1(t_1) = 0, \\ a_2(t_2) - b_2(t_2)e^{u_2^*} + \frac{S_2(t_2)}{e^{u_2^*}} + \mu^* D_2(t_2)e^{u_1^*-u_2^*} - \mu^* D_2(t_2) = 0, \\ -\mu^* \bar{d} + \bar{p}e^{u_1^*} - [\bar{q}+\bar{\beta}]e^{u_3^*} + \frac{\bar{S}_3}{e^{u_3^*}} = 0. \end{gathered} \tag{2.75}$$

By using the arguments of (2.19), (2.20), (2.27), (2.38), (2.44), (2.54), (2.59), (2.66), one can prove that

$$|u_i^*| \leq \max |M_1|, |M_2|, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|, \quad i = 1,2,3, \tag{2.76}$$

which implies that $\|u^*\| = |u_1^*| + |u_2^*| + |u_3^*| \leq 3\max\{M^*, |m_1^*|, |m_2^*|, |m_3^*|, |m_4^*|, |m_5^*|\} < M$. This contradicts the fact that $u^* \in \partial B_M(\mathbb{R}^3)$. Therefore, $H(u_1,u_2,u_3,\mu)$ is a homotopy.

Since

$$g(u) = \begin{pmatrix} \bar{a}_1 - \bar{D}_1 - \bar{b}_1 e^{u_1} - \bar{c}e^{u_3} + \bar{D}_1 e^{u_2-u_1} + \dfrac{\bar{S}_1}{e^{u_1}} \\ \bar{a}_2 - \bar{D}_2 - \bar{b}_2 e^{u_2} + \bar{D}_2 e^{u_1-u_2} + \dfrac{\bar{S}_2}{e^{u_2}} \\ -\bar{d} + \bar{p}e^{u_1} - [\bar{q}+\bar{\beta}]e^{u_3} + \dfrac{\bar{S}_3}{e^{u_3}} \end{pmatrix} = H(u_1,u_2,u_3,1), \tag{2.77}$$

$g(u) \neq 0$ for any $(u_1,u_2,u_3) \in \partial B_M(\mathbb{R}^3)$. Thus, the condition (ii) in Lemma 2.1 is satisfied. Next we show that condition (iii) also holds. It is easy to see that $H(u_1,u_2,u_3,0) = 0$ has a unique solution $v^* = (v_1^*,v_2^*,v_3^*)$, where v_1^*,v_2^*,v_3^* are the same as those in (2.71). Clearly, $\|v^*\| = |v_1^*| + |v_2^*| + |v_3^*| < M$, that is, $v^* \in B_M(\mathbb{R}^3)$. According to the invariance of homotopy, we obtain

$$\deg(g, B_M(\mathbb{R}^3)) = \deg(H(\cdot,1), B_M(\mathbb{R}^3)) = \deg(H(\cdot,0), B_M(\mathbb{R}^3)) = -1. \tag{2.78}$$

Therefore, all of the conditions required in Lemma 2.1 hold. According to Lemma 2.1, system (2.7) has one T-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$. It is easy to see that $(x_1^*(t), x_2^*(t), y^*(t))^T = (\exp[(u_1^*(t)], \exp[u_2^*(t)], \exp[u_3^*(t)])^T$ is a positive T-periodic solution of system (1.1). By the arguments similar to Claims 1–4, one can show

$$m_1 \leq u_i^*(t) \leq M_1 \quad (i = 1,2), \qquad m_2 \leq u_3^*(t) \leq M_2, \quad t \geq 0, \tag{2.79}$$

which implies

$$m_0 \leq x_i^*(t) \leq M_0 \quad (i = 1,2), \qquad \widetilde{m}_0 \leq y^*(t) \leq \widetilde{M}_0, \quad t \geq 0. \tag{2.80}$$

The proof is complete. □

Consider the special case of system (1.1) that $S_i(t) \equiv 0$, $i = 1,2,3$. In this case, by Theorem 2.2, we have the following.

COROLLARY 2.3. *In addition to* (H_1) *and* (H_2), *assume further that system (1.1) satisfies one of the following conditions:*
$(H_3)'$ $(a_i/b_i)^l > \bar{d}/\bar{p}$, $i = 1,2$;
$(H_4)'$ $((a_1 - D_1)/b_1)^l > \bar{d}/\bar{p}$.
Then system (1.1) has at least one positive T-periodic solution.

Remark 2.4. Corollary 2.3 greatly improves [15, Theorem 2.1] and [5, Theorem 1.1].

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