SUBORDINATION BY CONVEX FUNCTIONS

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For a fixed analytic function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ defined on the open unit disk and $\gamma < 1$, let $T_g(\gamma)$ denote the class of all analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying $\sum_{n=2}^{\infty} |a_n g_n| \le 1 - \gamma$. For functions in $T_g(\gamma)$, a subordination result is derived involving the convolution with a normalized convex function. Our result includes as special cases several earlier works.

1. Introduction

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \left(z \in \Delta := \{z \in \mathbb{C} : |z| < 1\}\right). \tag{1.1}$$

Let $S^*(\alpha)$ and $C(\alpha)$ be the usual classes of normalized starlike and convex functions of order α, respectively, and let $C := C(0)$. For $f(z)$ given by (1.1) and $g(z)$ by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{1.2}$$

the convolution (or Hadamard product) of f and g, denoted by $f * g$, is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n g_n z^n. \tag{1.3}$$

The function $f(z)$ is subordinate to the function $g(z)$, written as $f(z) \prec g(z)$, if there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

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Let $g(z)$ given by (1.2) be a fixed function, with $g_n \geq g_2 > 0$ $(n \geq 2)$, $\gamma < 1$, and let

$$T_g(\gamma) := \left\{ f(z) \in \mathcal{A} : \sum_{n=2}^{\infty} |a_n g_n| \leq 1 - \gamma \right\}. \tag{1.4}$$

The class $T_g(\gamma)$ includes as its special cases various other classes that were considered in several earlier works. In particular, for $\gamma = \alpha$ and $g_n = n - \alpha$, we obtain the class $TS^*(\alpha) := T_g(\gamma)$ that was introduced by Silverman [6]. Putting $\gamma = \alpha$ and $g_n = n(n - \alpha)$, we get $TC(\alpha) := T_g(\gamma)$. For these classes, Silverman [6] proved that $TS^*(\alpha) \subseteq S^*(\alpha)$ and $TC(\alpha) \subseteq C(\alpha)$.

By using convolution, Ruscheweyh [5] defined the operator

$$D^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} * f(z) \quad (\alpha > -1). \tag{1.5}$$

Let $R_\alpha(\beta)$ denote the class of functions $f(z)$ in $\mathcal{A}$ that satisfies the inequality

$$\Re \frac{D^{\alpha+1} f(z)}{D^\alpha f(z)} > \frac{\alpha + 2\beta}{2(\alpha+1)} \quad (\alpha \geq 0,\ 0 \leq \beta < 1,\ z \in \Delta). \tag{1.6}$$

Al-Amiri [1] called functions in this class as prestarlike functions of order α and type β. Let $H_\alpha(\beta)$ denote the class of functions $f(z)$ given by (1.1) whose coefficients satisfy the condition

$$\sum_{n=2}^{\infty} (2n + \alpha - 2\beta) C(\alpha, n) |a_n| \leq 2 + \alpha - 2\beta \quad (\alpha \geq 0,\ 0 \leq \beta < 1), \tag{1.7}$$

where

$$C(\alpha, n) := \prod_{k=2}^{n} \frac{(k + \alpha - 1)}{(n-1)!} \quad (n = 2, 3, \ldots). \tag{1.8}$$

Al-Amiri [1] proved that $H_\alpha(\beta) \subseteq R_\alpha(\beta)$. By taking $g_n = (2n + \alpha - 2\beta) C(\alpha, n)$ and $\gamma = 2\beta - 1 - \alpha$, we see that $H_\alpha(\beta) := T_g(\gamma)$.

For functions in the class $H_\alpha(\beta)$, Attiya [2] proved the following.

Theorem 1.1 [2, Theorem 2.1, page 3]. *If $f(z) \in H_\alpha(\beta)$ and $h(z) \in \mathcal{C}$, then*

$$\frac{(4 + \alpha - 2\beta)(1 + \alpha)}{2[\alpha + (2 + \alpha)(3 + \alpha - 2\beta)]} (f * h)(z) \prec h(z), \tag{1.9}$$

$$\Re(f(z)) > -\frac{\alpha + (2 + \alpha)(3 + \alpha - 2\beta)}{(4 + \alpha - 2\beta)(1 + \alpha)}. \tag{1.10}$$

The constant factor

$$\frac{(4 + \alpha - 2\beta)(1 + \alpha)}{2[\alpha + (2 + \alpha)(3 + \alpha - 2\beta)]} \tag{1.11}$$

in the subordination result (1.9) cannot be replaced by a larger number.

Owa and Srivastava [4] as well as Owa and Nishiwaki [3] studied the subclasses $\mathcal{M}^*(\alpha)$ and $\mathcal{N}^*(\alpha)$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\begin{aligned} &\sum_{n=2}^{\infty}\left[n-\lambda+|n+\lambda-2\alpha|\right]\left|a_n\right| \leq 2(\alpha-1) \quad (\alpha>1,\ 0\leq\lambda\leq 1),\\ &\sum_{n=2}^{\infty}n\left[n-\lambda+|n+\lambda-2\alpha|\right]\left|a_n\right| \leq 2(\alpha-1) \quad (\alpha>1,\ 0\leq\lambda\leq 1), \end{aligned} \tag{1.12}$$

respectively. These are special cases of $T_g(\gamma)$, with $g_n = n-\lambda+|n+\lambda-2\alpha|$, $\gamma = 3-2\alpha$, and $g_n = n(n-\lambda+|n+\lambda-2\alpha|)$, $\gamma = 3-2\alpha$, respectively. For the class $\mathcal{M}^*(\alpha)$, Srivastava and Attiya [8] proved the following.

Theorem 1.2 [8, Theorem 1, page 3]. *Let $f(z) \in \mathcal{M}^*(\alpha)$. Then for any function $h(z) \in C$ and $z \in \Delta$,*

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\left[2\alpha-\lambda+|2+\lambda-2\alpha|\right]}(f*h)(z) \prec h(z), \tag{1.13}$$

$$\Re\left(f(z)\right) > -\frac{2\alpha-\lambda+|2+\lambda-2\alpha|}{\left[(2-\lambda)+|2+\lambda-2\alpha|\right]}. \tag{1.14}$$

The constant factor

$$\frac{2-\lambda+|2+\lambda-2\alpha|}{2\left[2\alpha-\lambda+|2+\lambda-2\alpha|\right]} \tag{1.15}$$

in the subordination result (1.13) cannot be replaced by a larger number.

A similar result [8, Theorem 2, page 5] for $\mathcal{N}^*(\alpha)$ was also obtained.

In this article, Theorems 1.1 and 1.2 are unified for the class $T_g(\gamma)$. Relevant connections of our results with several earlier investigations are also indicated.

We need the following result on subordinating factor sequence to obtain our main result. Recall that a sequence $(b_n)_1^{\infty}$ of complex numbers is said to be a *subordinating factor sequence*, if for every convex univalent function $f(z)$ given by (1.1), then

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z). \tag{1.16}$$

Theorem 1.3 [9, Theorem 2, page 690]. *A sequence $(b_n)_1^{\infty}$ of complex numbers is a subordinating factor sequence if and only if*

$$\Re\left(1+2\sum_{n=1}^{\infty} b_n z^n\right) > 0. \tag{1.17}$$

2. Subordination with convex functions

We begin with the following subordination result.

THEOREM 2.1. *If $f(z) \in T_g(\gamma)$ and $h(z) \in C$, then*

$$\frac{g_2}{2(g_2+1-\gamma)}(f*h)(z) \prec h(z), \tag{2.1}$$

$$\Re(f(z)) > -\frac{g_2+1-\gamma}{g_2} \quad (z \in \Delta). \tag{2.2}$$

The constant factor

$$\frac{g_2}{2(g_2+1-\gamma)} \tag{2.3}$$

in the subordination result (2.1) cannot be replaced by a larger number.

Proof. Let $G(z) = z + \sum_{n=2}^{\infty} g_2 z^n$. Since $T_g(\gamma) \subseteq T_G(\gamma)$, our result follows if we prove the result for the class $T_G(\gamma)$. Let $f(z) \in T_G(\gamma)$ and suppose that

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in C. \tag{2.4}$$

In this case,

$$\frac{g_2}{2(g_2+1-\gamma)}(f*h)(z) = \frac{g_2}{2(g_2+1-\gamma)}\left(z + \sum_{n=2}^{\infty} c_n a_n z^n\right). \tag{2.5}$$

Observe that the subordination result (2.1) holds true if

$$\left(\frac{g_2}{2(g_2+1-\gamma)} a_n\right)_1^{\infty} \tag{2.6}$$

is a subordinating factor sequence (with of course, $a_1 = 1$). In view of Theorem 1.3, this is equivalent to the condition that

$$\Re\left\{1 + \sum_{n=1}^{\infty} \frac{g_2}{g_2+1-\gamma} a_n z^n\right\} > 0. \tag{2.7}$$

Since $g_n \geq g_2 > 0$ for $n \geq 2$, we have

$$\begin{aligned}
\Re\left\{1 + \frac{g_2}{g_2+1-\gamma} \sum_{n=1}^{\infty} a_n z^n\right\} &= \Re\left\{1 + \frac{g_2}{g_2+1-\gamma} z + \frac{1}{g_2+1-\gamma} \sum_{n=2}^{\infty} g_2 a_n z^n\right\} \\
&\geq 1 - \left\{\frac{g_2}{g_2+1-\gamma} r + \frac{1}{g_2+1-\gamma} \sum_{n=2}^{\infty} |g_2 a_n| r^n\right\} \\
&> 1 - \left\{\frac{g_2}{g_2+1-\gamma} r + \frac{1-\gamma}{g_2+1-\gamma} r\right\} > 0 \quad (|z| = r < 1).
\end{aligned} \tag{2.8}$$

Thus (2.7) holds true in Δ, and proves (2.1). The inequality (2.2) follows by taking $h(z) = z/(1-z)$ in (2.1).

Now consider the function

$$F(z) = z - \frac{1-\gamma}{g_2} z^2 \quad (\gamma < 1). \tag{2.9}$$

Clearly, $F(z) \in T_g(\gamma)$. For this function $F(z)$, (2.1) becomes

$$\frac{g_2}{2(g_2 + 1 - \gamma)} F(z) \prec \frac{z}{1-z}. \tag{2.10}$$

It is easily verified that

$$\min \left\{ \Re \left(\frac{g_2}{2(g_2 + 1 - \gamma)} F(z) \right) \right\} = -\frac{1}{2} \quad (z \in \Delta). \tag{2.11}$$

Therefore the constant

$$\frac{g_2}{2(g_2 + 1 - \gamma)} \tag{2.12}$$

cannot be replaced by any larger one. □

Corollary 2.2. *If $f(z) \in TS^*(\alpha)$ and $h(z) \in C$, then*

$$\frac{2-\alpha}{2(3-2\alpha)} (f * h)(z) \prec h(z), \qquad \Re(f(z)) > -\frac{3-2\alpha}{2-\alpha} \quad (z \in \Delta). \tag{2.13}$$

The constant factor

$$\frac{2-\alpha}{2(3-2\alpha)} \tag{2.14}$$

in the subordination result (2.13) cannot be replaced by a larger number.

Remark 2.3. The case $\alpha = 0$ in Corollary 2.2 was obtained by Singh [7].

Corollary 2.4. *If $f(z) \in TC(\alpha)$ and $h(z) \in C$, then*

$$\frac{2-\alpha}{5-3\alpha} (f * h)(z) \prec h(z), \qquad \Re(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)} \quad (z \in \Delta). \tag{2.15}$$

The constant factor

$$\frac{2-\alpha}{5-3\alpha} \tag{2.16}$$

in the subordination result (2.15) cannot be replaced by a larger one.

Remark 2.5. Theorem 1.1 is obtained by taking $\gamma = 2\beta - 1 - \alpha$ and

$$g_n = (2n + \alpha - 2\beta) \prod_{k=2}^{n} \frac{(k + \alpha - 1)}{(n-1)!} \quad (n = 2, 3 \ldots, \ \alpha > 0, \ 0 \le \beta < 1), \tag{2.17}$$

in Theorem 2.1. Similarly, putting $\gamma = 3 - 2\alpha$ and

$$g_n = n - \lambda + |n + \lambda - 2\alpha| \quad (n = 2,3\ldots,\ \alpha > 1,\ 0 \leq \lambda \leq 1) \tag{2.18}$$

in Theorem 2.1 yields Theorem 1.2. Finally, by taking $\gamma = 3 - 2\alpha$ and

$$g_n = n(n - \lambda + |n + \lambda - 2\alpha|) \quad (n = 2,3\ldots,\ \alpha > 1,\ 0 \leq \lambda \leq 1) \tag{2.19}$$

in Theorem 2.1, we get [8, Theorem 2, page 5].

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