ON SOLVABILITY OF GENERAL NONLINEAR VARIATIONAL-LIKE INEQUALITIES IN REFLEXIVE BANACH SPACES

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We introduce and study a new class of general nonlinear variational-like inequalities in reflexive Banach spaces. By applying a minimax inequality, we establish two existence and uniqueness theorems of solutions for the general nonlinear variational-like inequality.

1. Introduction

Recently, variational inequality theory has been extended and applied in various directions, see [6, 7, 8, 9, 10] and the references therein. In particular, Ding [1, 2, 3], and Ding and Tan [4] studied the existence of solutions for several nonlinear variational-like inequalities in reflexive Banach spaces.

In this paper, a new class of general nonlinear variational-like inequalities in reflexive Banach spaces are introduced. Utilizing a minimax inequality due to Ding and Tan [4], we provide some efficient conditions, which ensure the existence and uniqueness of solutions for the general nonlinear variational-like inequality. Our results improve and generalize many known results in the literature.

2. Preliminaries

Let D be a nonempty convex subset of a reflexive Banach space B with dual space B^* and let $\langle u,v\rangle$ be the dual pairing between $u \in B^*$ and $v \in B$. Let $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$ be mappings. Suppose that $a : B \times B \to (-\infty,+\infty)$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

(c1) $a(u,v) \geq c\|v\|^2$ for all $v \in B$;

(c2) $a(u,v) \leq d\|u\|\|v\|$ for all $u,v \in B$.

Clearly, $c \leq d$.

Let $f : D \to (-\infty,+\infty]$ be a real functional and let $z^* \in B^*$. We consider the following general nonlinear variational-like inequality problem: find $u \in D$ such that

$$\langle N(Tu,Au) - z^*, \eta(v,u)\rangle + a(u,v-u) \geq f(u) - f(v), \quad \forall v \in D. \tag{2.1}$$

We have the following special cases.

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(A) If $N(Tu,Au) = Tu - Au$, $a(u,v) \equiv 0$ for all $u,v \in D$, and $z^* = 0$, then the general nonlinear variational-like inequality (2.1) reduces to

$$\langle Tu - Au, \eta(v,u)\rangle \geq f(u) - f(v), \quad \forall v \in D, \tag{2.2}$$

which was introduced and studied by Ding [1].

(B) If $N(Tu,Au) = Tu - Au$ and $a(u,v) \equiv 0$ for all $u,v \in D$, then the general nonlinear variational-like inequality (2.1) reduces to

$$\langle Tu - Au - z^*, \eta(v,u)\rangle \geq f(u) - f(v), \quad \forall v \in D, \tag{2.3}$$

which was studied by Yao [10] in Hilbert spaces.

Definition 2.1. Let D be a nonempty subset of a reflexive Banach space B with dual space B^*. Let $T : D \to B^*$, $N : B^* \times B^* \to B^*$, and $\eta : D \times D \to B$ be mappings.

(1) T is said to be *Lipschitz continuous* with constant α if there exists a constant $\alpha > 0$ such that

$$\|Tu - Tv\| \leq \alpha\|u - v\|, \quad \forall u,v \in D. \tag{2.4}$$

(2) N is said to be *η-relaxed monotone* with constant γ with respect to T in the first argument if there exists a constant $\gamma > 0$ such that

$$\langle N(Tu,w) - N(Tv,w), \eta(u,v)\rangle \geq -\gamma\|u - v\|^2, \quad \forall u,v \in D,\ w \in B^*. \tag{2.5}$$

(3) N is said to be *η-strongly monotone* with constant ξ with respect to T in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Tu,w) - N(Tv,w), \eta(u,v)\rangle \geq \xi\|u - v\|^2, \quad \forall u,v \in D,\ w \in B^*. \tag{2.6}$$

(4) N is said to be *η-monotone* with respect to A in the second argument if

$$\langle N(w,Au) - N(w,Av), \eta(u,v)\rangle \geq 0, \quad \forall u,v \in D,\ w \in B^*. \tag{2.7}$$

(5) η is said to be Lipschitz continuous with constant δ if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leq \delta\|u - v\|, \quad \forall u,v \in D. \tag{2.8}$$

(6) η is said to be strongly monotone with constant τ if there exists a constant $\tau > 0$ such that

$$\langle u - v, \eta(u,v)\rangle \geq \tau\|u - v\|^2, \quad \forall u,v \in D. \tag{2.9}$$

Definition 2.2. Let D be a nonempty convex subset of a reflexive Banach space B and let $f : D \to (-\infty, +\infty]$ be a real functional.

(1) f is said to be *convex* if

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha)f(v), \quad \forall u,v \in D,\ \alpha \in [0,1]. \tag{2.10}$$

(2) f is said to be *lower semicontinuous* on D if for each $\alpha \in (-\infty,+\infty]$, the set $\{u \in D : f(u) \leq \alpha\}$ is closed in D.

Definition 2.3. Let D be a nonempty subset of a reflexive Banach space B with dual space B^* and let $T : D \to B^*$ and $\eta : D \times D \to B$ be two mappings. T and η are said to have 0-*diagonally concave relation* with respect to $z^* \in B^*$ if the functional $\varphi : D \times D \to (-\infty,+\infty]$ defined by $\varphi(u,v) = \langle Tu - z^*, \eta(u,v)\rangle$ is 0-*diagonally concave* in v, that is, for any finite set $\{v_1,\dots,v_m\} \in D$ and for any $u = \sum_{i=1}^m \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$,

$$\sum_{i=1}^{m} \lambda_i \varphi(u,v_i) \leq 0. \tag{2.11}$$

Remark 2.4. It is easy to see that, if for each $u \in D$, $\eta(u,u) = 0$ and the functional $v \mapsto \langle Tu - z^*, \eta(u,v)\rangle$ is concave, then the mappings T and η have the 0-diagonally concave relation with respect to z^* on D.

Lemma 2.5 [4]. *Let D be a nonempty convex subset of a topological vector space and let $\varphi : D \times D \to [-\infty,+\infty]$ be such that*

(a) *for each $u \in D, v \mapsto \varphi(u,v)$ is lower semicontinuous on each nonempty compact subset of D,*

(b) *for each nonempty finite set $\{v_1,\dots,v_m\} \in D$ and for any $u = \sum_{i=1}^m \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$, $\min_{1\leq i\leq m} \varphi(u,v_i) \leq 0$,*

(c) *there exist a nonempty compact convex subset X_0 of D and a nonempty compact subset K of D such that for each $v \in D - K$, there is $u \in co(X_0 \cup \{v\})$ with $\varphi(u,v) > 0$.*

Then there exists $\hat{v} \in K$ such that $\varphi(u,\hat{v}) \leq 0$ for all $u \in D$.

3. Existence and uniqueness theorems

In this section, we use the minimax inequality technique due to Ding and Tan [4] to prove the existence and uniqueness theorems of solutions for the general nonlinear variational-like inequality (2.1).

Theorem 3.1. *Let D be a nonempty closed convex subset of a reflexive Banach space B with dual space B^*. Assume that $\eta : D \times D \to B$ is Lipschitz continuous with constant δ, for each $v \in D$, $\eta(\cdot,v)$ is continuous on D, and $\eta(v,u) = -\eta(u,v)$ for all $v,u \in D$. Suppose that $a : B \times B \to (-\infty,+\infty)$ is a coercive continuous bilinear form and $f : D \to (-\infty,+\infty]$ is a proper convex lower semicontinuous functional with* $\operatorname{int}(\operatorname{dom} f) \cap D \neq \emptyset$. *Let T, A : $D \to B^*$ and $N : B^* \times B^* \to B^*$ be continuous mappings, let N be η-strongly monotone with constant α with respect to T in the first argument and η-monotone with respect to A in the second argument. Assume that N and η have the 0-diagonally concave relation with respect to $z^* \in B^*$. Then the general nonlinear variational-like inequality (2.1) has a unique solution $\hat{u} \in D$.*

Proof. Define a functional $\varphi : D \times D \to (-\infty,+\infty]$ by

$$\varphi(v,u) = \langle z^* - N(Fu,Gu), \eta(v,u)\rangle - a(u,v-u) + f(u) - f(v), \quad \forall u,v \in D. \tag{3.1}$$

Since T, A, N, a, and η are continuous and f is lower semicontinuous, it follows that for each $v \in D$, the functional $u \mapsto \varphi(v,u)$ is weakly lower semicontinuous on D. We claim that φ satisfies the condition (b) of Lemma 2.5. If it is false, there exist a finite set $\{v_1,\dots,v_m\} \subset D$ and $u = \sum_{i=1}^m \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that $\varphi(v_i,u) > 0$ for all $i = 1,\dots,m$, that is,

$$\langle z^* - N(Tu,Au), \eta(v_i,u)\rangle - a(u,v_i) + f(u) - f(v_i) > 0 \tag{3.2}$$

for all $i = 1,\dots,m$. It follows that

$$\sum_{i=1}^{m} \lambda_i \langle z^* - N(Tu,Au), \eta(v_i,u)\rangle > \sum_{i=1}^{m} \lambda_i a(u,v_i) - f(u) + \sum_{i=1}^{m} \lambda_i f(v_i) \geq 0, \tag{3.3}$$

which contradicts the condition that N and η have the 0-diagonally concave relation with respect to z^*. Therefore, the condition (b) of Lemma 2.5 holds. Since f is proper convex lower semicontinuous, it follows from [5] that its subdifferential $\partial f(v) \neq \emptyset$ for all $v \in \operatorname{int}(\operatorname{dom} f)$. It is easy to see that $f(u) \geq f(v^*) + \langle r, u - v^*\rangle$ for all $v^* \in \operatorname{int}(\operatorname{dom} f) \cap D$, $r \in \partial f(v^*)$, and $u \in B$. For any fixed $v^* \in \operatorname{int}(\operatorname{dom} f) \cap D$, set

$$Q = (\alpha + c)^{-1}\left[\delta(\|N(Tv^*,Av^*)\| + \|z^*\|) + d\|v^*\| + \|r\|\right] \tag{3.4}$$

and $K = \{u \in D : \|u - v^*\| \leq Q\}$. Then K and $D_0 = \{v^*\}$ are both weakly compact convex subsets of D. It follows that for each $u \in D - K$,

$$\begin{aligned}
\varphi(v^*,u) &= \langle z^* - N(Tu,Au), \eta(v^*,u)\rangle - a(u,v^* - u) + f(u) - f(v^*) \\
&\geq \langle z^*, \eta(v^*,u)\rangle + \langle N(Tu,Au) - N(Tv^*,Au), \eta(u,v^*)\rangle \\
&\quad + \langle N(Tv^*,Au) - N(Fv^*,Gv^*), \eta(u,v^*)\rangle \\
&\quad - \langle N(Fv^*,Gv^*), \eta(v^*,u)\rangle \\
&\quad + a(v^* - u, v^* - u) - a(v^*, v^* - u) + \langle r, u - v^*\rangle \\
&\geq \|u - v^*\|\left[(\alpha + c)\|u - v^*\| - \delta(\|N(Tv^*,Av^*)\| + \|z^*\|) - d\|v^*\| - \|r\|\right] > 0,
\end{aligned} \tag{3.5}$$

that is, the condition (c) of Lemma 2.5 holds. By Lemma 2.5, there exists $\hat{u} \in D$ such that $\varphi(v,\hat{u}) \leq 0$ for all $v \in D$, that is,

$$\langle N(T\hat{u},A\hat{u}) - z^*, \eta(v,\hat{u})\rangle + a(\hat{u}, v - \hat{u}) \geq f(\hat{u}) - f(v), \quad \forall v \in D. \tag{3.6}$$

Now we prove that $\hat{u}$ is a unique solution of the general nonlinear variational-like inequality (2.1). Suppose that u_1 and u_2 are two solutions of the general nonlinear variational-like inequality (2.1). It follows that

$$\begin{aligned}
\langle N(Fu_1,Gu_1) - z^*, \eta(u_2,u_1)\rangle + a(u_1, u_2 - u_1) \geq f(u_1) - f(u_2), \\
\langle N(Fu_2,Gu_2) - z^*, \eta(u_1,u_2)\rangle + a(u_2, u_1 - u_2) \geq f(u_2) - f(u_1).
\end{aligned} \tag{3.7}$$

Using $\eta(u,v) = -\eta(v,u)$ for all u and $v \in D$, (3.7), we deduce that

$$\begin{aligned} 0 &\geq \langle N(Tu_1,Au_1) - N(Tu_2,Au_2),\eta(u_1,u_2)\rangle + a(u_1 - u_2,u_1 - u_2) \\ &\geq (\alpha + c)\|u_1 - u_2\|^2 \geq 0, \end{aligned} \tag{3.8}$$

which means that $u_1 = u_2$ and $\hat{u}$ is the unique solution of the general nonlinear variational-like inequality (2.1). This completes the proof. □

Theorem 3.2. *Let D, B, B^*, η, a, f, T, and N be as in Theorem 3.1, let $A : D \to B^*$ be Lipschitz continuous with constant γ, and let $N : B^* \times B^* \to B^*$ be η-relaxed monotone with constant α with respect to T in the first argument and Lipschitz continuous with constant β in the second argument. Assume that N and η have the 0-diagonally concave relation with respect to $z^* \in B^*$ and $c > \alpha + \beta\gamma\delta$. Then the general nonlinear variational-like inequality (2.1) has a unique solution $\hat{u} \in D$.*

Proof. Let

$$Q = (c - \alpha - \beta\gamma\delta)^{-1}\left[\delta(\|N(Tv^*,Av^*)\| + \|z^*\|) + d\|v^*\| + \|r\|\right] \tag{3.9}$$

and $K = \{u \in D : \|u - v^*\| \leq Q\}$. It follows from the proof of Theorem 3.1 that

$$\begin{aligned} \varphi(v^*,u) &= \langle z^*,\eta(v^*,u)\rangle + \langle N(Tu,Au) - N(Tv^*,Au),\eta(u,v^*)\rangle \\ &\quad - \langle N(Tv^*,Au) - N(Tv^*,Av^*),\eta(v^*,u)\rangle \\ &\quad - \langle N(Tv^*,Av^*),\eta(v^*,u)\rangle \\ &\quad + a(v^* - u,v^* - u) - a(v^*,v^* - u) + \langle r,u - v^*\rangle \\ &\geq \|u - v^*\|\big[(c - \alpha - \beta\gamma\delta)\|u - v^*\| \\ &\qquad - \delta(\|N(Tv^*,Av^*)\| + \|z^*\|) - d\|v^*\| - \|r\|\big] > 0. \end{aligned} \tag{3.10}$$

The rest of the proof follows precisely as in the proof of Theorem 3.1. This completes the proof. □

Remark 3.3. Theorems 3.1 and 3.2 improve Ding's [1, Theorems 3.1 and 3.2] and Yao's [10, Theorem 3.1].

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