ON REDUCIBILITY OF SOME OPERATOR SEMIGROUPS AND ALGEBRAS ON LOCALLY CONVEX SPACES

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Received 9 March 2005 and in revised form 20 October 2005

A generalization of some results from normed spaces, concerning reducibility and triangularizability of semigroups and algebras of operators, to locally convex spaces is given.

1. Introduction

Let X be a complex Hausdorff locally convex space. A system of seminorms P inducing the topology on X will be called a calibration. We denote by $\mathcal{P}(X)$ the collection of all calibrations on X. For a given $P \in \mathcal{P}(X)$ let $P = \{p_\alpha : \alpha \in \Delta\}$, where Δ is some index set and for each $\alpha \in \Delta$ denote $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Let us denote by $\mathcal{L}(X)$ the set of all linear continuous operators on X, by $\mathcal{K}(X)$ the set of compact operators on X ($T \in \mathcal{K}(X)$ if there exists a neighbourhood U_γ such that $T(U_\gamma)$ is a relatively compact set), by $\mathcal{F}(X)$ the set of all finite-rank operators and by $\mathcal{L}B(X)$ the set of all locally bounded operators (there is some neighbourhood U_γ such that $T(U_\gamma)$ is bounded). The topology of bounded convergence on $\mathcal{L}(X)$ and on X' will be denoted by τ_b. By X'_b we will denote the topological space (X', τ_b). We will denote by $\mathcal{R}(T)$ the range of T and by $\mathcal{N}(T)$ the null space of T. For a given $T \in \mathcal{L}(X)$ the number $\lambda \in \mathbb{C}$ is in the resolvent set of T if and only if $(\lambda I - T)^{-1}$ exists in $\mathcal{L}(X)$. The spectrum $\sigma(T)$ is the complement of the resolvent set and by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ we denote the spectral radius of T. An operator T is quasinilpotent if $\sigma(T) = \{0\}$. A closed subspace M in X is an *invariant subspace* of an operator T if $T(M) \subseteq M$. A collection of linear operators is *reducible* if it has a common nontrivial invariant subspace and is *irreducible* otherwise. If a family $\mathcal{A} \subset \mathcal{L}(X)$ is an algebra, it is irreducible if and only if it is transitive, that is, the set $\mathcal{A}x := \{Tx : T \in \mathcal{A}\}$ is dense in X for each $x \neq 0$. For $P \in \mathcal{P}(X)$ and $p_\alpha \in P$ let J_α denote the null space of p_α and X_α the quotient space X/J_α. It is a normed space with the norm $\|x_\alpha\|_\alpha := p_\alpha(x)$, where $x_\alpha = x + J_\alpha$. Let $T \in \mathcal{L}(X)$ be such that $T(J_\alpha) \subset J_\alpha$ then the corresponding operator T_α on X_α is well defined by $T_\alpha(x_\alpha) = Tx + J_\alpha$.

2. The results

Lemma 2.1. *Let X be a locally convex space and $\mathcal{A}$ a transitive τ_b-closed algebra of continuous operators on X which contains a nonzero finite-rank operator. Then there exists a*

International Journal of Mathematics and Mathematical Sciences 2005:24 (2005) 3951–3961
DOI: 10.1155/IJMMS.2005.3951

τ_b-closed subspace Φ in X'_b such that

(i) *$\mathscr{A}$ contains all rank-one operators of the form $x \otimes f$, $x \in X$, $f \in \Phi$,*

(ii) *if $f(z) = 0$ for each $f \in \Phi$, then $z = 0$.*

Proof. Let $F \in \mathscr{A}$ be a nonzero finite-rank operator. Denote $E_1 = \mathscr{R}(F)$ and $\mathscr{A}_1 := F\mathscr{A}$. It is clear that each $T \in \mathscr{A}_1$ maps E_1 to E_1 and the restriction $\mathscr{A}_1|_{E_1}$ is a transitive algebra of operators on a finite-dimensional space. By Burnside's theorem [11] it follows that $\mathscr{A}_1|_{E_1} = \mathscr{L}(E_1)$. Especially, there is some $A_0 \in \mathscr{A}$ such that $FA_0|_{E_1}$ has rank one, hence FA_0F is a rank-one operator in $\mathscr{A}$ of the form $x_0 \otimes f_0$, $x_0 \in X$, and $f_0 \in X'$. Choose arbitrary nonzero $x \in X$. Since $\mathscr{A}x_0$ is dense in X, there is some net of operators $\{A_\delta\} \subset \mathscr{A}$ such that $A_\delta x_0 \to x$. For any chosen seminorm q_α^M defining the topology τ_b we have

$$q_\alpha^M\big((A_\delta x_0) \otimes f_0 - x \otimes f_0\big) = \sup_{y \in M} |f_0(y)|\, p_\alpha(A_\delta x_0 - x) \le c \cdot p_\alpha(A_\delta x_0 - x). \tag{2.1}$$

Since the right-hand side tends to zero, the same holds for the left-hand side, thus $x \otimes f_0 \in \mathscr{A}$. Let us define $\Phi := \{f \in X' : x \otimes f \in \mathscr{A}, \text{ for all } x \in X\}$. Choose $\{f_\delta\} \subset \Phi$ a net which is τ_b-convergent to some f, and $x \in X$ arbitrary, then for any seminorm q_α^M we have

$$q_\alpha^M(x \otimes f_\delta - x \otimes f) = \sup_{y \in M} |(f_\delta - f)(y)|\, p_\alpha(x). \tag{2.2}$$

Since $\mathscr{A}$ is τ_b-closed we have $x \otimes f \in \mathscr{A}$, hence $f \in \Phi$. Thus, Φ is a nontrivial τ_b-closed subspace satisfying (i). To verify (ii), let $f(z) = 0$ for each $f \in \Phi$. Choose a nonzero $f_0 \in \Phi$ and for each $A \in \mathscr{A}$ define $f_1 := A'f_0$, where A' is the adjoint operator of A. Since $x \otimes f_1 = x \otimes A'f_0 = (x \otimes f_0)A \in \mathscr{A}$, for any $x \in X$, we have $f_1 \in \Phi$, hence $f_1(z) = 0$, that is $f_0(Az) = 0$ for each $A \in \mathscr{A}$. If there were $z \neq 0$, then f_0 would be zero on a dense set $\mathscr{A}z$ and consequently identically zero, which is a contradiction. □

Corollary 2.2. *Let X be a semireflexive locally convex space and $\mathscr{A}$ a transitive τ_b-closed algebra of continuous operators which contains a nonzero finite-rank operator. Then $\mathscr{A}$ contains all finite-rank operators.*

Proof. It is sufficient to show that $\mathscr{A}$ contains all rank-one operators and this will be in case $\Phi = X'$. If there were $\Phi \neq X'$, then by the Hahn-Banach theorem there would be some nonzero $F \in (X'_b)'$, such that $F|_\Phi = 0$. Since X is semireflexive, there is some nonzero $y \in X$ such that $F(f) = f(y)$ for all $f \in X'$ and then $f(y) = F(f) = 0$ for all $f \in \Phi$. By (ii) in the previous lemma then $y = 0$, which is a contradiction. □

A linear operator T is called *nuclear* if it can be written in the form

$$Tx = \sum_{j=1}^{\infty} \lambda_j c_j(x) a_j, \quad x \in X, \tag{2.3}$$

where $\{c_j\}$ is an equicontinuous sequence in X', $\{\lambda_j\} \in l_1$, and $\{a_j\}$ is a sequence contained in an absolutely convex bounded set B in X, such that $X_B := \bigcup\{nB : n \in \mathbb{N}\}$ is a complete normed subspace in X with respect to the Minkowski's functional of the set B. (see, e.g., [8]). It is easy to see that the family of nuclear operators is an ideal in $\mathscr{L}(X)$ and that each nuclear operator is also compact.

Corollary 2.3. *Let X be a semireflexive locally convex space and $\mathcal{A}$ a transitive algebra of continuous operators such that $\overline{(\mathcal{A},\tau_b)}$ contains a nonzero finite-rank operator. Then $\overline{(\mathcal{A},\tau_b)}$ contains all nuclear operators.*

Proof. By Corollary 2.2, $\overline{(\mathcal{A},\tau_b)} \supset \mathcal{F}(X)$, since each nuclear operator can be τ_b-approximated by finite-rank operators by [1], the conclusion follows. □

For a given set $\mathcal{M}$ of compact operators let us denote by $\widetilde{\mathcal{M}}$ the set of all $A \in \mathcal{K}(X)$ which are τ_b-limits of some sequence $\{A_n\} \subset \mathcal{M}$. By [5, Proposition 1] it follows that if X is a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X, then $\widetilde{\mathcal{S}}$ is a semigroup too.

Proposition 2.4. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators. If $A \in \mathcal{S}$ is such that $r(A) \neq 0$ then there exists a sequence $\{n_i\}$ of integers such that one of the following assertions holds:*

(a) *$A^{n_i} \xrightarrow{\tau_b} E$, where E is idempotent, or*

(b) *$\alpha_i A^{n_i} \xrightarrow{\tau_b} E$, for some scalar sequence $\{\alpha_i\}$, where E is nilpotent.*

In both cases $E \in \widetilde{\mathbb{R}^+\mathcal{S}}$ and is of finite rank.

Proof. Following the first part of the proof of [6, Proposition 3] we can find a sequence of operators from $\mathcal{S}$ with the above property. □

Theorem 2.5. *Let X be a barreled locally convex space and $\mathcal{A}$ a τ_b-closed transitive algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. Then $\mathcal{A} \cap \mathcal{F}(X)$ is a nontrivial transitive algebra.*

Proof. Define $\mathcal{C} = \mathcal{A} \cap \mathcal{K}(X)$, then it is an ideal in $\mathcal{A}$ and by [6, Lemma 5] it is transitive too. Let K be a nonzero operator in $\mathcal{C}$. Then by Lomonosov's theorem for locally convex spaces [3] there is some $A \in \mathcal{C}$ such that $r(AK) \neq 0$. By Proposition 2.4, $\widetilde{\mathcal{C}} \cap \mathcal{F}(X) \neq \{0\}$. Since $\mathcal{A}$ is τ_b-closed, it is easy to see that $\widetilde{\mathcal{C}} = \mathcal{C}$. Clearly, $\mathcal{A} \cap \mathcal{F}(X) = \mathcal{C} \cap \mathcal{F}(X)$ is an ideal in $\mathcal{A}$ and it is transitive too. □

Corollary 2.6. *Let X be a barreled semireflexive locally convex space and $\mathcal{A}$ a τ_b-closed transitive algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. Then $\mathcal{A}$ contains all finite-rank operators.*

Proof. By Theorem 2.5 and Corollary 2.2, the conclusion follows. □

Corollary 2.7. *Let X be a barreled semireflexive locally convex space and $\mathcal{A}$ a transitive τ_b-closed algebra in $\mathcal{L}(X)$ which contains a nonzero compact operator. If X has the property that each compact operator on X is τ_b-limit of a net of finite-rank operators, then $\mathcal{A}$ contains all compact operators.*

Proposition 2.8. *Let X be a semireflexive locally convex space, $\mathcal{S}$ a semigroup in $\mathcal{F}(X)$, and ϕ a nontrivial τ_b-continuous linear functional on $\mathcal{F}(X)$. If ϕ is identically zero on $\mathcal{S}$, then $\mathcal{S}$ is reducible.*

Proof. Let us suppose that $\mathcal{S}$ is irreducible. Then the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is also irreducible and the same holds for $\overline{(\mathcal{A},\tau_b)}$. By Corollary 2.2, $\overline{(\mathcal{A},\tau_b)} \supset \mathcal{F}(X)$. Since ϕ is equal to zero on $\mathcal{S}$, it is equal to zero also on $\mathcal{F}(X)$, which is a contradiction. □

With respect to the strong topology τ_s on $\mathcal{L}(X)$ the following theorem is proven in [8].

Theorem 2.9. *Let X be a locally convex space and $\mathcal{A}$ a transitive algebra in $\mathcal{L}(X)$, such that $\overline{(\mathcal{A},\tau_s)}$ contains a nontrivial compact operator. Then $\mathcal{A}$ is τ_s-dense in $\mathcal{L}(X)$.*

Corollary 2.10. *Let X be a locally convex space and A a nonscalar operator commuting with a nonzero compact operator. Then A has a nontrivial hyperinvariant subspace.*

Proof. Denote by $\mathcal{A} := (A)'$ the commutant of A. Clearly, it is an algebra and let us prove that it is τ_s-closed. Choose any net $\{B_\delta\}$ in $\mathcal{A}$ which is strongly convergent to some B. Then for any seminorm q_x^α for the strong topology one has

$$\begin{aligned} q_x^\alpha(BA-AB) &= p_\alpha((BA-AB)x) \le p_\alpha((B-B_\delta)Ax)+p_\alpha(A(B-B_\delta)x) \\ &\le q_y^\alpha(B-B_\delta)+c_\alpha q_x^\beta(B-B_\delta), \end{aligned} \tag{2.4}$$

where $y = Ax$. Since the right-hand side is arbitrary small, the left-hand side is zero. Since q_x^α is arbitrary, we have $BA-AB=0$. If $\mathcal{A}$ were transitive, then by the above theorem, it would be equal to $\mathcal{L}(X)$ and consequently $A=\lambda I$ for some complex number λ, which is a contradiction. □

Corollary 2.11. *Let X be a locally convex space and $A,B \in \mathcal{L}(X)$ two commuting operators, where B is nonscalar and commutes with a nonzero compact operator. Then A has a nontrivial invariant subspace.*

Proof. By Corollary 2.10, B has a hyperinvariant subspace which is invariant for A. □

Corollary 2.12. *Let X be a locally convex space and $\mathcal{S}$ a semigroup of operators such that $\overline{(\mathcal{S},\tau_s)}$ contains a nonzero compact operator and ϕ a nontrivial τ_s-continuous linear functional on $\mathcal{L}(X)$ such that $\phi|_{\mathcal{S}}=0$. Then $\mathcal{S}$ is reducible.*

Proof. Let $\mathcal{S}$ be irreducible, then the algebra $\mathcal{A}$ generated by $\mathcal{S}$ is also irreducible and then by Theorem 2.9, it is strongly dense in $\mathcal{L}(X)$. Clearly, ϕ is equal to zero also on $\mathcal{A}$ and thus on $\mathcal{L}(X)$, which is a contradiction. □

Corollary 2.13. *Let X be a locally convex space and $\mathcal{E}$ a commutative family of compact operators on X. Then $\mathcal{E}$ is reducible.*

Proof. Choose a nonzero $A \in \mathcal{E}$, by Corollary 2.10 it has a nontrivial hyperinvariant subspace which is then an invariant subspace for $\mathcal{E}$. □

Let ϕ be a functional on a semigroup $\mathcal{S} \subset \mathcal{L}(X)$. By [11] ϕ is *permutable* on a family $\mathcal{E} \subset \mathcal{S}$ if for any $A_1,A_2,\dots,A_n \in \mathcal{E}$ and any permutation τ of $\{1,2,\dots,n\}$ we have $\phi(A_1,A_2,\dots,A_n) = \phi(A_{\tau(1)},A_{\tau(2)},\dots,A_{\tau(n)})$.

Proposition 2.14. *Let X be a semireflexive locally convex space and ϕ a nontrivial τ_b-continuous linear functional on $\mathcal{F}(X)$. Let $\mathcal{E}$ be a family of finite-rank operators such that ϕ is permutable on $\mathcal{E}$. Then $\mathcal{E}$ is reducible.*

Proof. Since ϕ is permutable on $\mathcal{E}$, it is also permutable on the algebra $\mathcal{A}$ generated by $\mathcal{E}$. In view of Corollary 2.13 we can assume that $\mathcal{E}$ is noncommutative. Then there are

$A, B \in \mathscr{E}$ such that $C := AB - BA \neq 0$. Denote by $\mathscr{J}$ the ideal in $\mathscr{A}$ generated by C. Clearly, $\phi(SCT) = 0$ for each $S, T \in \mathscr{A}$, consequently, $\phi|_{\mathscr{J}} = 0$. Hence, by Proposition 2.8, $\mathscr{J}$ is reducible, then by [6], $\mathscr{A}$ is reducible and $\mathscr{E}$ is reducible too. □

Corollary 2.15. *Let X be a semireflexive locally convex space, $\mathscr{S}$ a semigroup of finite-rank operators, and ϕ a nontrivial τ_b-continuous linear functional on $\mathscr{F}(X)$. Then $\mathscr{S}$ is reducible if one of the following conditions holds:*

(i) *ϕ is multiplicative on $\mathscr{S}$,*

(ii) *ϕ is constant on $\mathscr{S}$.*

For fixed nonzero $x_0 \in X$, $f \in X'$ and a subspace $\mathscr{M}$ in $\mathscr{L}(X)$ let us define (as in [11]) the so-called *coordinate functional* by the relation $\phi(T) := f(Tx_0)$, $T \in \mathscr{M}$. For this class of functionals we do not need semireflexivity of the space.

Lemma 2.16. *Let X be a locally convex space, then the coordinate functional ϕ is τ_b-continuous on $\mathscr{L}(X)$ if and only if f is continuous on X.*

Proof. If f is continuous, then $|\phi(T)| = |f(Tx_0)| \leq cp_\beta(Tx_0) \leq cq_\beta^M(T)$, where M is an arbitrary bounded set in X containing x_0. Let ϕ be τ_b-continuous on $\mathscr{L}(X)$. Choose any $x \in X$. By the Hahn-Banach theorem there is some $g \in X'$ such that $g(x_0) = 1$ and let $S = x \otimes g$, hence $Sx_0 = x$. By continuity of ϕ there exist a seminorm q_α^M and $c > 0$ such that $|f(x)| = |f(Sx_0)| = |\phi(S)| \leq cq_\alpha^M(S) = c\sup_{y\in M} p_\alpha(g(y)x) \leq c_1 p_\alpha(x)\sup_{y\in M} p_\gamma(y) = d_\gamma p_\alpha(x)$ for some constant $d_\gamma > 0$. □

Proposition 2.17. *Let X be a locally convex space, $\mathscr{S}$ a semigroup in $\mathscr{L}(X)$, and ϕ a τ_b-continuous coordinate functional on $\mathscr{L}(X)$. Then $\mathscr{S}$ is reducible if one of the following conditions holds:*

(i) *ϕ is constant on $\mathscr{S}$,*

(ii) *ϕ is multiplicative on $\mathscr{S}$.*

The proof is the same as for the normed space (see [11, Lemma 8.2.8]).

Corollary 2.18. *Let X be a locally convex space, $\mathscr{E}$ a noncommutative family in $\mathscr{L}(X)$, and ϕ a τ_b-continuous coordinate functional which is permutable on $\mathscr{E}$, then $\mathscr{E}$ is reducible.*

Proof. Since permutability is inherited by passing to an algebra (see [11, page 28]), we can assume that $\mathscr{E}$ is an algebra. Let us choose A, B in $\mathscr{E}$ such that $C := AB - BA \neq 0$. Then it is easy to see that ϕ is equal to zero on ideal $\mathscr{J}$ generated by C and by the previous proposition $\mathscr{J}$ is reducible, hence $\mathscr{E}$ is reducible. □

Lemma 2.19. *Let X be a locally convex space and $A, B \in \mathscr{L}(X)$ such that $\sigma(AB)$ is bounded. Then $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$ and $r(AB) = r(BA)$.*

Proof. It is easy to see that if for $\lambda \neq 0$ there exists $C := (\lambda I - AB)^{-1} \in \mathscr{L}(X)$. Then we also have that $(\lambda I - BA)^{-1} = \lambda^{-1}(I + BCA) \in \mathscr{L}(X)$. Hence also $\sigma(BA)$ is bounded and both equalities follow. □

A locally convex space is called *H-locally convex* if its topology can be defined by a calibration P such that each $p_\alpha \in P$ is generated by a semiscalar product: $p_\alpha^2(x) = (x, x)_\alpha$

(see, e.g., [10]). In H-locally convex spaces the trace functional is well-defined on nuclear operators and a generalization to these spaces of Lidskii's theorem holds, which says that the trace of nuclear operator is equal to the sum of its eigenvalues, (see [7]). By this theorem and by the above lemma we also have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for each pair of nuclear operators A and B acting on an H-locally convex space.

Theorem 2.20. *Let $\mathcal{S}$ be a family of nuclear operators on a barreled H-locally convex space. Then $\mathcal{S}$ is triangularizable if and only if the trace functional is permutable on $\mathcal{S}$.*

Proof. We can verify as in [11, Lemma 2.1.14] that the permutability of the trace functional is equivalent to the condition $\operatorname{tr}(ABC) = \operatorname{tr}(ACB)$, for all $A,B,C \in \mathcal{S}$. Let $\mathcal{S}$ be triangularizable. Then for the diagonal elements the following relations hold:

$$d_j(ABC) = d_j(A)d_j(B)d_j(C) = d_j(ACB); \quad A,B,C \in \mathcal{S} \tag{2.5}$$

(see [9]), where it is also shown that the nonzero eigenvalues coincide with nonzero diagonal elements for each operator of $\mathcal{S}$. Thus, for any $A,B,C \in \mathcal{S}$ we have

$$\operatorname{tr}(ABC) = \sum \lambda_j(ABC) = \sum d_j(ABC) = \sum d_j(ACB) = \operatorname{tr}(ACB). \tag{2.6}$$

Let the trace be permutable on $\mathcal{S}$. Then it is also permutable on the algebra $\mathcal{A}$ generated by $\mathcal{S}$. Hence $\operatorname{tr}(A(BC - CB)) = 0$, $A,B,C \in \mathcal{A}$. Taking $A = (BC - CB)^{n-1}$, for $n \in \mathbb{N}$ we obtain $\operatorname{tr}((BC - CB)^n) = 0$, $n \in \mathbb{N}$. Denote $T := BC - CB$, since $T \in \mathcal{K}(X)$, we have $\lambda_j(T^n) = \lambda_j(T)^n$ for each j (see [12]). Hence, by Lidskii's theorem we have

$$\sum_j \lambda_j(T)^n = \sum_j \lambda_j(T^n) = \operatorname{tr}(T^n) = 0, \quad n \in \mathbb{N}. \tag{2.7}$$

As in [4] it follows that $\lambda_j(T) = 0$ for each j, thus, $\sigma(BC - CB) = \{0\}$ for each pair $B,C \in \mathcal{A}$ and by [5, Theorem 2], $\mathcal{A}$ is triangularizable and the same holds for $\mathcal{S}$. □

Corollary 2.21. *Let $\mathcal{S}$ be a family of nuclear operators on a barreled H-locally convex space. Then $\mathcal{S}$ is triangularizable if the trace functional satisfies one of the following conditions:*

(i) it is multiplicative on $\mathcal{S}$,

(ii) it is constant on $\mathcal{S}$.

The following result is a generalization of the so-called "downsizing lemma" from [11].

Lemma 2.22. *Let X be a barreled locally convex space, $\mathcal{S}$ a semigroup of compact operators on X and $\mathcal{P}$ a property on $\mathcal{S}$ such that*

(i) *each subsemigroup in $\mathcal{S}$ has the property $\mathcal{P}$,*

(ii) *$\mathcal{S}|_{X_0}$, has the property $\mathcal{P}$, where $X_0 = \overline{\operatorname{span}\{\mathcal{R}(S),\, S \in \mathcal{S}\}}$,*

(iii) *$\widetilde{\mathbb{R}^+\mathcal{S}}$ has the property $\mathcal{P}$.*

Let $\mathcal{S}$ be irreducible. Then there exist a natural number $k \geq 2$ and an idempotent operator $E \in \mathcal{L}(X)$ of rank k, such that $\mathcal{S}$ contains a subsemigroup $\mathcal{S}_0$ with properties: $\mathcal{S}_0 = E\mathcal{S}_0$, $\mathcal{S}_0|_{\mathcal{R}(E)}$ is irreducible in $\mathcal{L}(\mathbb{C}^k)$ and it has the property $\mathcal{P}$. Moreover, if $\min\{\operatorname{rank}(F) : F \in \widetilde{\mathbb{R}^+\mathcal{S}}\} > 1$, *the operator E can be chosen from $\widetilde{\mathbb{R}^+\mathcal{S}}$.*

Proof. By [6, Theorem 3], not all operators in $\mathcal{S}$ are quasinilpotent, hence by Proposition 2.4, $\widetilde{\mathbb{R}^+\mathcal{S}}$ contains a nonzero finite-rank operator. By [6, Proposition 2], $\widehat{\mathcal{S}} := \widetilde{\mathbb{R}^+\mathcal{S}}$ is also a semigroup. Then the proof is the same as for the normed space (see [11, Lemma 8.2.13]), where we take $\widehat{\mathcal{S}}$ instead of $\mathcal{S}$. □

Proposition 2.23. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X. If $\mathcal{S}^k$ is triangularizable for some $k \in \mathbb{N}$, then $\mathcal{S}$ is triangularizable.*

Proof. If $\mathcal{S}^k = \{0\}$, then by [6, Theorem 3], $\mathcal{S}$ is reducible. Let $\mathcal{S}^k \neq \{0\}$ and let $\mathcal{S}^k$ be triangularizable, then $\mathcal{S}$ is reducible since $\mathcal{S}^k$ is an ideal in $\mathcal{S}$. The triangularizability of a family of compact operators $\mathcal{S}^k$ is inherited by quotients which follows in the same manner as in the proof of [11, Theorem 7.3.9] for the normed space. Then applying the triangularization lemma [5] the triangularization of $\mathcal{S}$ follows. □

Theorem 2.24. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X. If $AB - BA$ is quasinilpotent for every $A, B \in \mathcal{S}$, then $\mathcal{S}$ is triangularizable.*

Proof. By [5, Lemma 5], the quasinilpotency is inherited by quotients for compact operators. So, by triangularizing lemma it suffices to prove the reducibility of the semigroup $\mathcal{S}$. Let us verify the conditions of Lemma 2.22. The condition (i) is obvious. Denoting by X_0 the closed span of $\{\mathcal{R}(S) : S \in \mathcal{S}\}$, then for $A, B \in \mathcal{S}$ it easy to see that $A|_{X_0}B|_{X_0} - B|_{X_0}A|_{X_0} = (AB - BA)|_{X_0}$ and it is clear that $(AB - BA)|_{X_0}$ is also quasinilpotent, hence (ii) holds. By [5, Theorem 1], quasinilpotency is inherited by $\widetilde{\mathbb{R}^+\mathcal{S}}$ and so (iii) holds. If $\mathcal{S}$ were irreducible, then by Lemma 2.22, there would exist a subsemigroup $\mathcal{S}_0$ of $\mathcal{S}$ such that $\mathcal{S}_0|_{\mathcal{R}(E)}$ would be irreducible in $\mathcal{L}(\mathbb{C}^k)$, which is impossible [11, Theorem 4.4.12]. □

Let $\mathcal{E}$ be a semigroup in $\mathcal{L}B(X)$. It is known (see, e.g., [12]) that the spectrum for each $T \in \mathcal{L}B(X)$ is bounded. We say that *the spectrum is submultiplicative* on $\mathcal{E}$ if $\sigma(AB) \subset \sigma(A)\sigma(B) = \{\lambda\mu : \lambda \in \sigma(A),\ \mu \in \sigma(B)\}$ for all $A, B \in \mathcal{E}$.

Theorem 2.25. *Let X be an infinite-dimensional barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators on X with the property that the spectrum is submultiplicative on $\mathcal{S}$. Then $\mathcal{S}$ is reducible.*

Proof. Let us denote by $\widehat{\mathcal{S}} := \widetilde{\mathbb{R}^+\mathcal{S}}$. Suppose that $\mathcal{S}$ is irreducible, then $\widehat{\mathcal{S}}$ is irreducible too. By [6, Proposition 3] there exists nonzero finite-rank idempotent operator $E \in \widehat{\mathcal{S}}$ which has minimal rank m. Denoting $\mathcal{S}_0 = \widehat{\mathcal{S}}E\widehat{\mathcal{S}}$, this is an ideal in $\widehat{\mathcal{S}}$ and all operators in $\mathcal{S}_0$ have a rank equal to m or 0. Thus, $\mathcal{S}_0$ is irreducible and then the rest of the proof is the same as for the normed space (see [11, Theorem 8.3.5]), where we take $\mathcal{S}_0$ instead of $\mathcal{S}$. □

Theorem 2.26. *Let X be an infinite-dimensional barreled locally convex space and $\mathcal{S}$ a semigroup of compact operators with the following property: $\sigma(S) \subset \{0, 1\}$ for every $S \in \mathcal{S}$ and if $1 \in \sigma(ST)$, for $S, T \in \mathcal{S}$, let $1 \in \sigma(S) \cap \sigma(T)$. Then $\mathcal{S}$ is reducible.*

Proof. We can assume that X is not normable (see [11, Theorem 8.3.8]). Since each $A \in \mathcal{S}$ is locally bounded, it has 0 in his spectrum [12]. With the above assumption the submultiplicativity of spectrum on $\mathcal{S}$ follows, so $\mathcal{S}$ is reducible by the previous theorem. □

Theorem 2.27. *Let X be a barreled locally convex space and $\mathcal{S}$ a semigroup of compact idempotent operators. Then $\mathcal{S}$ is triangularizable.*

Proof. In view of [11, Theorem 2.3.5] let X be infinite-dimensional. A quotient of an idempotent is, clearly, idempotent operator and a quotient of compact operator is again compact (see [5]). So, by the triangularization lemma, it suffices to prove the reducibility. Clearly, for an idempotent S and $\lambda \neq 0, \neq 1$, there exists $(\lambda I - S)^{-1} = I/\lambda - S/(\lambda(1-\lambda)) \in \mathcal{L}(X)$, hence $\sigma(S) \subset \{0,1\}$. If $1 \in \sigma(ST)$ for $S, T \in \mathcal{S}$, then S and T are nonzero idempotent, hence $1 \in \sigma(S) \cap \sigma(T)$. Thus, the conditions of the preceding theorem are fulfilled and the reducibility of $\mathcal{S}$ follows. □

For a semiball U_γ, $\gamma \in \Delta$, let us denote by $\mathcal{L}_\gamma(X)$ the family of all continuous linear operators T on X, for which $T(U_\gamma)$ is a bounded set. Clearly, this is a subspace and left ideal in $\mathcal{L}(X)$. For each $T \in \mathcal{L}_\gamma(X)$ we have $T(U_\gamma) \subset \lambda_\gamma U_\gamma$ for some $\lambda_\gamma > 0$, hence $T(J_\gamma) \subset J_\gamma$, thus, operator T_γ is well defined on X_γ. For some fixed $T \in \mathcal{L}_\gamma(X)$ the convergence of some sequence in $\mathcal{L}(X)$ is inherited to the operator sequence on the quotient space X_γ in the following sense.

Lemma 2.28. *Let X be a locally convex space, $T \in \mathcal{L}_\gamma(X)$ for some $\gamma \in \Delta$ and $\{S_n\} \subset \mathcal{L}(X)$ a sequence which is τ_b-convergent to some S in $\mathcal{L}(X)$. Then $\{(S_nT)_\gamma\}$ is convergent to $(ST)_\gamma$ with respect to the norm $\|\cdot\|_\gamma$ in X_γ.*

Proof. Since $M := T(U_\gamma)$ is a bounded set, we have $\|(S_nT)_\gamma - (ST)_\gamma\|_\gamma = \sup_{x \in U_\gamma} p_\gamma((S_nT - ST)x) = \sup_{y \in M} p_\gamma((S_n - S)y) = q_\gamma^M(S_n - S) \to 0$, as $n \to \infty$. □

It is well known that in a normed space the spectral radius is continuous on the set of compact operators (see, e.g., [11]), but this is not the case for general locally convex spaces. Let us take as an example $X = s$, the space of all real sequences $\{x_n\}$ with the topology generated by seminorms $P = \{p_m : m \in \mathbb{N}\}$, where $p_m(x) = \sup\{|x_j| : j \leq m\}$, $x \in X$ and a sequence of operators $\{T_n\}$ defined by $T_n(x_1, x_2, \ldots) = (0, 0, \ldots, x_n, 0, 0, \ldots)$. It is easy to see that all T_n are compact and $T_n \xrightarrow{\tau_b} T$, where $T = 0$. Hence, $r(T_n) = 1$ for all $n \in \mathbb{N}$, but $r(T) = 0$. We will prove the continuity of spectral radius in a special case.

Lemma 2.29. *Let X be a locally convex space, $T \in \mathcal{K}(X)$ and $\{S_n\}$ a sequence of continuous operators which is τ_b-convergent to $S \in \mathcal{L}(X)$. Then*

$$r(S_nT) \longrightarrow r(ST), \quad r(TS_n) \longrightarrow r(TS), \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

Proof. Since T is locally bounded, there is some $\gamma \in \Delta$ such that $T \in \mathcal{L}_\gamma(X)$ and also $AT \in \mathcal{L}_\gamma(X)$ for each $A \in \mathcal{L}(X)$. The corresponding operator T_γ is also compact on X_γ and for the spectral radius we have $r(T) = r(T_\gamma)$ (see [2]). Then, by Lemma 2.28 and by continuity of the spectral radius for a sequence of compact operators acting on a normed space, we obtain $r(S_nT) = r((S_nT)_\gamma) \to r((ST)_\gamma) = r(ST)$, as $n \to \infty$. By Lemma 2.19 we have also $r(TS_n) \to r(TS)$, as $n \to \infty$. □

Lemma 2.30. *Let X be a locally convex space, $\mathcal{S}$ a semigroup in $\mathcal{L}(X)$ and A, B some nonzero members in $\mathcal{S}$ such that $B\mathcal{S}A = \{0\}$. Then $\mathcal{S}$ is reducible.*

Proof. If $\mathcal{S}\mathcal{R}(A) = \{0\}$, then it is easy to see that $\overline{\mathcal{R}(A)}$ is a nontrivial invariant subspace for $\mathcal{S}$. If $\mathcal{S}\mathcal{R}(A) \neq \{0\}$, then the closed span of this set is a nontrivial invariant subspace for $\mathcal{S}$. □

Lemma 2.31. *Let X be a locally convex space and $\mathcal{S}$ an irreducible semigroup of compact operators on X. If spectral radius is submultiplicative on $\mathcal{S}$ then no nonzero product $\widetilde{S}\widetilde{T}$, where $\widetilde{S}, \widetilde{T} \in \widetilde{\mathcal{S}}$, is quasinilpotent.*

Proof. Let us denote $\mathcal{J} = \{\widetilde{S}\widetilde{T} : \widetilde{S}, \widetilde{T} \in \widetilde{\mathcal{S}},\ r(\widetilde{S}\widetilde{T}) = 0\}$. This is an outer ideal of $\mathcal{S}$. Indeed, choose any product $\widetilde{S}\widetilde{T} \in \mathcal{J}$ and $C \in \mathcal{S}$, then there exist two sequences $\{S_n\}$ and $\{T_m\}$ in $\mathcal{S}$ such that $S_n \xrightarrow{\tau_b} \widetilde{S}$ and $T_m \xrightarrow{\tau_b} \widetilde{T}$. For each pair of operators S_n, T_m we have

$$r(CS_nT_m) \le r(C)r(S_nT_m); \quad n,m \in \mathbb{N}. \tag{2.9}$$

Using Lemma 2.29 twice we obtain $r(C\widetilde{S}\widetilde{T}) \le r(C)r(\widetilde{S}\widetilde{T}) = 0$, hence, $C\widetilde{S}\widetilde{T} \in \mathcal{J}$. Since $r(\widetilde{S}\widetilde{T}C) = r(C\widetilde{S}\widetilde{T})$, also $\widetilde{S}\widetilde{T}C \in \mathcal{J}$. If $\mathcal{J} \neq \{0\}$, by [6, Theorem 3], it would be reducible and then by [6, Lemma 5], $\mathcal{S}$ would be reducible too. Thus, $\mathcal{J} = \{0\}$. □

Theorem 2.32. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on a barreled locally convex space. If spectral radius is submultiplicative on $\mathcal{S}$, then it is multiplicative on $\mathcal{S}$.*

Proof. Since the spectral radius is homogenous for the nonnegative scalars, one can suppose $\mathbb{R}^+\mathcal{S} = \mathcal{S}$. By [6], $\widetilde{\mathcal{S}}$ is again a semigroup. Let us prove that it has no quasinilpotent element. Suppose, to the contrary, that there is a nonzero quasinilpotent operator $\widetilde{T}$ in $\widetilde{\mathcal{S}}$. Then also $r(\widetilde{T}^2) = 0$, hence by the previous lemma, $\widetilde{T}^2 = 0$. Consequently $r(\widetilde{T}\widetilde{S}\widetilde{T}) = r(\widetilde{S}\widetilde{T}^2) = 0$ for any $\widetilde{S} \in \widetilde{\mathcal{S}}$. Thus $\widetilde{T}\widetilde{\mathcal{S}}\widetilde{T} = \{0\}$, hence, by Lemma 2.30, $\widetilde{\mathcal{S}}$ is reducible, which is a contradiction. Choose any $A, B \in \mathcal{S}$, where we can assume $r(A) = r(B) = 1$. By Proposition 2.4 there exist two nonzero finite-rank idempotents $E, F \in \widetilde{\mathcal{S}}$ such that $A^{n_i} \xrightarrow{\tau_b} E$ and $B^{m_k} \xrightarrow{\tau_b} F$ for some sequences of integers $\{n_i\}$ and $\{m_k\}$. Let us prove that $EF \neq 0$. If $EF = 0$, then $r(F\widetilde{S}E) = r(\widetilde{S}EF) = 0$ and by the above lemma it would be $F\widetilde{S}E = 0$ for each $\widetilde{S} \in \widetilde{\mathcal{S}}$, hence $F\widetilde{\mathcal{S}}E = \{0\}$ and by Lemma 2.30, $\widetilde{\mathcal{S}}$ would be reducible. In the sequel, let us prove the following inequality:

$$r(EF^2E) \le r(EF)^2. \tag{2.10}$$

For each pair of operators from the sequences defining E and F we have by assumption $r(A^{n_i}B^{2m_k}A^{n_i}) \le r(A^{n_i}B^{m_k})r(B^{m_k}A^{n_i})$ and by Lemma 2.29 we obtain the above inequality. Similarly, we have $r(A^{n_i}B^{m_k}) = r(ABB^{m_k-1}A^{n_i-1}) \le r(AB)r(A)^{n_i-1}r(B)^{m_k-1} = r(AB)$ and by Lemma 2.29 we obtain

$$r(EF) \le r(AB). \tag{2.11}$$

From the idempotency of E and F and by the inequality (2.10) it follows that:

$$r(EF) = r(E^2F^2) = r(EF^2E) \le r(EF)^2. \tag{2.12}$$

Thus, we have $1 \le r(EF) \le r(AB) \le r(A)r(B) = 1$, and the multiplicativity of the spectral radius follows. □

THEOREM 2.33. *Let X be a barreled locally convex space, $\mathcal{S}$ a semigroup of compact operators on X such that each $S \in \mathcal{S}$ is a nonnegative scalar multiple of an idempotent operator and let spectral radius be submultiplicative on $\mathcal{S}$. Then $\mathcal{S}$ is triangularizable.*

Proof. Let us prove that $\mathcal{S}$ is reducible. Denote $\mathcal{S}_0 = \{S/r(S) : S \in \mathcal{S},\, S \ne 0\} \cup \{0\}$. Clearly, $\mathcal{S}_0$ is reducible if and only if $\mathcal{S}$ is reducible. Suppose that $\mathcal{S}$ is irreducible. Then, by Theorem 2.32 the spectral radius is multiplicative on $\mathcal{S}$. Consequently, $\mathcal{S}_0$ is a semigroup of compact idempotents. By Theorem 2.27, $\mathcal{S}_0$ is reducible. Thus, $\mathcal{S}$ is reducible and by triangularization lemma it is triangularizable. □

In view of Lemma 2.19 it is easy to see that the spectral radius is permutable on a semigroup $\mathcal{S}$ if and only if $r(ABC) = r(ACB)$ for all $A, B \in \mathcal{S}$.

THEOREM 2.34. *Let X be a locally convex space and $\mathcal{S}$ a semigroup of compact operators on X. Then spectral radius is submultiplicative on $\mathcal{S}$ if and only if it is permutable on $\mathcal{S}$.*

Proof. We will use the property $r(T) = r(T_\gamma)$ for $T \in \mathcal{L}_\gamma(X)$, $\gamma \in \Delta$ (see [2]). With no loss of generality we may assume that the calibration $P \in \mathcal{P}(X)$ is *directed*, that is for each $p_\alpha, p_\beta \in P$ there is some $p_\gamma \in P$ such that $p_\alpha \le p_\gamma$ and $p_\beta \le p_\gamma$. Let r be permutable on $\mathcal{S}$. Choose any $A, B \in \mathcal{S}$. Since they are locally bounded and P is directed, there exists $p_\gamma \in P$ such that $A, B \in \mathcal{L}_\gamma(X)$. Denote by $\mathcal{S}_0^\gamma$ the semigroup generated by A_γ, B_γ. By [6, Lemma 1] spectral radius is also permutable on $\mathcal{S}_0^\gamma$ and by [11, Theorem 8.6.3] it is submultiplicative and then $r(AB) = r(A_\gamma B_\gamma) \le r(A_\gamma)r(B_\gamma) = r(A)r(B)$. Let r be submultiplicative on $\mathcal{S}$. For any $A, B, C \in \mathcal{S}$ there is some $p_\gamma \in P$ such that $A, B, C \in \mathcal{L}_\gamma(X)$, then on the semigroup $\mathcal{S}_1^\gamma$ generated by A_γ, B_γ, and C_γ the submultiplicativity implies the permutability of r and similarly as above we obtain $r(ABC) = r(ACB)$. □

Question. What are the conditions on a family of compact operators on a locally convex space yielding the continuity of the spectral radius on this family?

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