NOTES ON DOUBLE INEQUALITIES OF MATHIEU'S SERIES

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Using the integral expression of Mathieu's series and some integral and analytic inequalities involving periodic functions and the generating function of Bernoulli numbers, we present several new inequalities and estimates for Mathieu's series and generalize Mathieu's series. Two open problems are proposed.

1. Introduction

In 1890, Mathieu defined $S(r)$ in [10] as

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{\left(n^2 + r^2\right)^2}, \quad r > 0, \tag{1.1}$$

and conjectured that $S(r) < 1/r^2$. We call formula (1.1) Mathieu's series.

This conjecture was proved in 1952 by Berg in [2]. Since then, various papers appeared providing interesting new inequalities involving $S(r)$. Please refer to references listed in this paper.

In [9], Makai proved

$$\frac{1}{r^2 + 1/2} < S(r) < \frac{1}{r^2}. \tag{1.2}$$

Alzer et al. in [1] obtained

$$\frac{1}{x^2 + 1/(2\zeta(3))} < S(x) < \frac{1}{x^2 + 1/6}, \tag{1.3}$$

where ζ denotes the zeta function. The inequalities in (1.3) are sharp: the constants $1/(2\zeta(3))$ and $1/6$ are the best possibe.

The integral form of Mathieu's series (1.1) was given in [3, 4] by

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x}{e^x - 1} \sin(rx)\, dx. \tag{1.4}$$

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In this paper, using the integral expression (1.4) of Mathieu's series and certain inequalities involving periodic functions and the generating function of Bernoulli numbers, some new inequalities and estimates for Mathieu's series are established and Mathieu's series is generalized. At the end, two open problems are proposed.

2. General results

In this section, we will establish several general theorems and inequalities involving periodic functions and then obtain some general inequalities for Mathieu's series.

Lemma 2.1. *Let $\psi(x)$ be an integrable function satisfying $\psi(x) = -\psi(x+T)$, where T is a given positive number, and $\psi(x) \ge 0$ for $x \in [0,T]$, let $f(x)$ and $g(x)$ be two integrable functions on $[0,2T]$ such that*

$$f(x) - g(x) \ge f(x+T) - g(x+T) \tag{2.1}$$

on $[0,T]$. Then

$$\int_0^{2T} \psi(x) f(x)dx \ge \int_0^{2T} \psi(x) g(x)dx. \tag{2.2}$$

Proof. By easy computation, it is deduced that

$$\begin{aligned}
&\int_0^{2T} \psi(x)[f(x)-g(x)]dx \\
&\quad = \int_0^{T} \psi(x)[f(x)-g(x)]dx + \int_T^{2T} \psi(x)[f(x)-g(x)]dx \\
&\quad = \int_0^{T} \psi(x)[f(x)-g(x)]dx + \int_0^{T} \psi(x+T)[f(x+T)-g(x+T)]dx \\
&\quad = \int_0^{T} \psi(x)\{[f(x)-g(x)]-[f(x+T)-g(x+T)]\}dx \\
&\quad \ge 0.
\end{aligned} \tag{2.3}$$

The proof is complete. □

Corollary 2.2. *Let $\psi(x) \not\equiv 0$ be an integrable periodic function with period $2T > 0$ satisfying $\psi(x) = -\psi(x+T)$ and $\psi(x) \ge 0$ for $x \in [0,T]$. If $f(x)$ is an integrable function such that $f(x) \ge f(x+T)$ on $[0,T]$, then*

$$\int_0^{2T} \psi(x) f(x)dx \ge 0. \tag{2.4}$$

Corollary 2.3. *Let $f(x)$ be an integrable function such that $f(x) \ge f(x+\pi)$ on $[0,\pi]$, then*

$$\int_0^{2\pi} f(x)\sin x\, dx \ge 0. \tag{2.5}$$

As a direct consequence of Lemma 2.1, we have the following theorem.

Theorem 2.4. *Let Φ_1 and Φ_2 be two integrable functions such that $x/(e^x-1)-\Phi_1(x)$ and $\Phi_2(x)-x/(e^x-1)$ are both increasing. Then, for any positive number r,*

$$\frac{1}{r}\int_0^\infty \Phi_2(x)\sin(rx)dx \le \sum_{n=1}^\infty \frac{2n}{(n^2+r^2)^2} \le \frac{1}{r}\int_0^\infty \Phi_1(x)\sin(rx)dx. \tag{2.6}$$

Proof. The function $\psi(x)=\sin(rx)$ has a period $2\pi/r$, and $\psi(x)=-\psi(x+\pi/r)$.

Since $f(x)=x/(e^x-1)-\Phi_1(x)$ is increasing, for any $\alpha>0$, we have $f(x+\alpha)\ge f(x)$. Therefore, from Corollary 2.2, we obtain

$$\begin{gathered}\int_{2k\pi/r}^{2(k+1)\pi/r}\left[\frac{x}{e^x-1}-\Phi_1(x)\right]\sin(rx)dx \le 0,\\ \int_{2k\pi/r}^{2(k+1)\pi/r}\frac{x}{e^x-1}\sin(rx)dx \le \int_{2k\pi/r}^{2(k+1)\pi/r}\Phi_1(x)\sin(rx)dx.\end{gathered} \tag{2.7}$$

Then, from formula (1.4), we have

$$\begin{aligned} S(r) &= \frac{1}{r}\sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r}\frac{x}{e^x-1}\sin(rx)dx \\ &\le \frac{1}{r}\sum_{k=0}^\infty \int_{2k\pi/r}^{2(k+1)\pi/r}\Phi_1(x)\sin(rx)dx \\ &= \frac{1}{r}\int_0^\infty \Phi_1(x)\sin(rx)dx. \end{aligned} \tag{2.8}$$

The right-hand side of inequality (2.6) follows.

Similar arguments yield the left-hand side of inequality (2.6). □

3. The first concrete result

Using Theorem 2.4 obtained in the previous section, now we will give the first concrete estimate for Mathieu's series by monotonicity of difference between a function related to the exponential function and the generating function $x/(e^x-1)$ of Bernoulli numbers.

Proposition 3.1. *The function*

$$g(x)=\frac{x}{e^x-1}-\frac{x^2}{e^{3x}-e^x} \tag{3.1}$$

is decreasing with $x>0$.

Proof. The proof follows from elementary analysis and standard argument. □

Theorem 3.2. *For any positive number $r>0$,*

$$\sum_{n=1}^\infty \frac{2n}{(n^2+r^2)^2} \ge \frac{2(r^2-3)}{(1+r^2)^3}+\frac{\pi^3}{8r}\operatorname{sech}^2\left(\frac{\pi r}{2}\right)\tanh\left(\frac{\pi r}{2}\right). \tag{3.2}$$

Proof. In [15, page 356], the following formula is given:

$$\int_0^\infty \frac{x^{2m}\sin(ax)}{e^{(2n+1)\alpha x}-e^{(2n-1)\alpha x}}\,dx=(-1)^m\frac{\partial^{2m}}{\partial a^{2m}}\left[\frac{\pi}{4\alpha}\tanh\frac{a\pi}{2\alpha}-\sum_{\nu=1}^{n}\frac{a}{a^2+(2\nu-1)^2\alpha^2}\right], \tag{3.3}$$

where $\alpha>0$ and $m,n=0,1,2,\ldots$. If $n=0$ in formula (3.3), then the summation terms are omitted.

Therefore, we have

$$\begin{aligned}\int_0^\infty \frac{x^2\sin(rx)}{e^{3x}-e^x}\,dx&=-\frac{\partial^2}{\partial r^2}\left[\frac{\pi}{4}\tanh\frac{\pi r}{2}-\frac{r}{r^2+1}\right]\\&=\frac{1}{8(1+r^2)^3}\left[16r(r^2-3)+\pi^3(r^2+1)^3\operatorname{sech}^2\left(\frac{\pi r}{2}\right)\tanh\left(\frac{\pi r}{2}\right)\right].\end{aligned} \tag{3.4}$$

From Theorem 2.4 and Proposition 3.1, inequality (3.2) follows. □

4. The second concrete result

In this section, by an inequality relating to the generating function $x/(e^x-1)$ of Bernoulli numbers, using the periodicity of the sine function, and reducing the integral expression (1.4) of Mathieu's series to another series, we will formulate another meaningful estimate to Mathieu's series.

Proposition 4.1. *For $x>0$,*

$$\frac{1}{e^x}<\frac{x}{e^x-1}<\frac{1}{e^{x/2}}. \tag{4.1}$$

Proof. This follows from standard argument. □

Theorem 4.2. *For any positive number $r>0$,*

$$\frac{4(1+r^2)\left(e^{-\pi/r}+e^{-\pi/(2r)}\right)-4r^2-1}{\left(e^{-\pi/r}-1\right)(1+r^2)(1+4r^2)}\le S(r)\le\frac{(1+4r^2)\left(e^{-\pi/r}-e^{-\pi/(2r)}\right)-4(1+r^2)}{\left(e^{-\pi/r}-1\right)(1+r^2)(1+4r^2)}. \tag{4.2}$$

Proof. For $r>0$, using (1.4), by direct calculation, we have

$$S(r)=\frac{1}{r}\sum_{k=0}^{\infty}\left[\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{x\sin(rx)}{e^x-1}\,dx+\int_{(2k+1)\pi/r}^{(2k+2)\pi/r}\frac{x\sin(rx)}{e^x-1}\,dx\right]. \tag{4.3}$$

The inequality (4.1) gives us

$$\begin{aligned}
\frac{r(1+e^{-\pi/r})}{(1+r^2)(1-e^{-2\pi/r})} &= \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{e^x}dx \le \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{x\sin(rx)}{e^x-1}dx \\
&\le \sum_{k=0}^{\infty}\int_{2k\pi/r}^{(2k+1)\pi/r}\frac{\sin(rx)}{e^{x/2}}dx = \frac{4r(1+e^{-\pi/(2r)})}{(1+4r^2)(1-e^{-\pi/r})}, \\
-\frac{4r(e^{-\pi/r}+e^{-\pi/(2r)})}{(1+4r^2)(1-e^{-\pi/r})} &= \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{e^{x/2}}dx \le \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{x\sin(rx)}{e^x-1}dx \\
&\le \sum_{k=0}^{\infty}\int_{(2k+1)\pi/r}^{2(k+1)\pi/r}\frac{\sin(rx)}{e^x}dx = -\frac{r(e^{-2\pi/r}+e^{-\pi/r})}{(1+r^2)(1-e^{-2\pi/r})}.
\end{aligned} \tag{4.4}$$

Substituting (4.4) into (4.3) yields

$$\begin{aligned}
&\frac{4e^{-\pi/r}+4r^2e^{-\pi/r}+4e^{-\pi/(2r)}+4r^2e^{-\pi/(2r)}-4r^2-1}{(e^{-\pi/r}-1)(1+r^2)(1+4r^2)} \\
&= \frac{1+e^{-\pi/r}}{(1+r^2)(1-e^{-2\pi/r})} - \frac{4(e^{-\pi/r}+e^{-\pi/(2r)})}{(1+4r^2)(1-e^{-\pi/r})} \\
&\le S(r) \\
&\le \frac{4(1+e^{-\pi/(2r)})}{(1+4r^2)(1-e^{-\pi/r})} - \frac{e^{-2\pi/r}+e^{-\pi/r}}{(1+r^2)(1-e^{-2\pi/r})} \\
&= \frac{e^{-\pi/r}+4r^2e^{-\pi/r}-4e^{-\pi/(2r)}-4r^2e^{-\pi/(2r)}-4-4r^2}{(e^{-\pi/r}-1)(1+r^2)(1+4r^2)}.
\end{aligned} \tag{4.5}$$

The proof is complete. □

Remark 4.3. When $0<r<0.83273\ldots$, the upper bound in (4.2) is better than that in (1.3). In fact, straightforward computation yields

$$\lim_{r\to 0}\frac{(1+4r^2)(e^{-\pi/r}-e^{-\pi/(2r)})-4(1+r^2)}{(e^{-\pi/r}-1)(1+r^2)(1+4r^2)} = 4 < 6 = \lim_{r\to 0}\frac{1}{r^2+1/6}. \tag{4.6}$$

When $0<r<2.9002\ldots$, the lower bound in (4.2) is positive, and then is useful. But, it is not better than that in (1.3).

5. The third concrete result

In this section, we will give another result using an approach similar to that in the previous section.

Theorem 5.1. *For any positive number $r>0$,*

$$\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^2} < \frac{1}{r}\int_0^{\pi/r}\frac{x}{e^x-1}\sin(rx)dx < \frac{1+\exp(-\pi/(2r))}{r^2+1/4}. \tag{5.1}$$

Proof. Straightforward computation yields

$$\begin{aligned}
&\int_0^\infty \frac{x}{e^x-1}\sin(rx)dx-\int_0^{\pi/r}\frac{x}{e^x-1}\sin(rx)dx\\
&\quad=\sum_{k=1}^\infty\int_{k\pi/r}^{(k+1)\pi/r}\frac{x}{e^x-1}\sin(rx)dx\\
&\quad=\sum_{i=1}^\infty\left(\int_{2i\pi/r}^{(2i+1)\pi/r}+\int_{(2i-1)\pi/r}^{2i\pi/r}\right)\frac{x}{e^x-1}\sin(rx)dx\\
&\quad=\sum_{i=1}^\infty\left(\int_0^\pi+\int_{-\pi}^0\right)\frac{(2i\pi+t)/r}{\exp((2i\pi+t)/r)-1}\sin(2i\pi+t)\frac{dt}{r}\\
&\quad=-\frac{1}{r}\sum_{k=1}^\infty\int_0^\pi\left[\frac{[(2k-1)\pi+t]/r}{\exp([(2k-1)\pi+t]/r)-1}-\frac{(2k\pi+t)/r}{\exp((2k\pi+t)/r)-1}\right]\sin t\,dt.
\end{aligned}\tag{5.2}$$

It is easy to see that the function $(e^t-1)/t$ is strictly increasing on $(0,+\infty)$, therefore, for all $t>0$, we have

$$\frac{[(2k-1)\pi+t]/r}{\exp([(2k-1)\pi+t]/r)-1}>\frac{(2k\pi+t)/r}{\exp((2k\pi+t)/r)-1}.\tag{5.3}$$

Then, from inequality (4.1), we have

$$\int_0^\infty \frac{x}{e^x-1}\sin(rx)dx<\int_0^{\pi/r}\frac{x}{e^x-1}\sin(rx)dx<\int_0^{\pi/r}e^{-x/2}\sin(rx)dx=\frac{r(1+\exp(-\pi/2r))}{r^2+1/4}.\tag{5.4}$$

Inequality (5.1) follows from combining (5.4) with (1.4). □

Remark 5.2. The monotonicity and convexity of the function $(e^t-1)/t$ can be deduced from those of the function $(b^t-a^t)/t$. For details, please refer to [6, 7, 8, 11, 12, 13, 14].

Remark 5.3. If $r>1.57482\ldots$, the upper bound in (5.1) is better than that in (4.2). When $r<1.574816\ldots$, the upper bound in (5.1) is not better than that in (4.2). When $0<r<0.734821\ldots$, the upper bound in (5.1) is better than that in (1.3).

6. Open problems

Now we would like to propose two open problems as follows.

Open Problem 6.1. Let

$$S(r,t,\alpha)=\sum_{n=1}^\infty\frac{2n^{\alpha/2}}{(n^\alpha+r^2)^{t+1}}\tag{6.1}$$

for $t>0$, $r>0$, and $\alpha>0$. Can one obtain an integral expression of $S(r,t,\alpha)$? Give some sharp inequalities for the series $S(r,t,\alpha)$.

In [15, page 356], the following formula is given

$$\int_0^\infty \frac{x^{2m}\sin(ax)}{e^{2n\alpha x}-e^{(2n-2)\alpha x}}dx = (-1)^m \frac{\partial^{2m}}{\partial a^{2m}}\left[\frac{\pi}{4\alpha}\coth\frac{a\pi}{2\alpha}-\frac{1}{2a}-\sum_{\nu=1}^{n-1}\frac{a}{a^2+(2\nu)^2\alpha^2}\right], \tag{6.2}$$

where $a>0$, $\alpha>0$, $m=0,1,2,\ldots$, and $n=1,2,\ldots$. If $n=1$ in formula (6.2), then the summation terms are omitted.

Remark 6.2. One can also find the formulae (3.3) and (6.2) in other handbooks on integral formulae.

Open Problem 6.3. Find suitable ranges of numbers α, m, and n such that

$$\begin{gathered}\frac{x}{e^x-1}-\frac{x^{2m}}{e^{(2n+1)\alpha x}-e^{(2n-1)\alpha x}},\quad \alpha>0 \text{ and } m,n=0,1,2,\ldots,\\ \frac{x}{e^x-1}-\frac{x^{2m}}{e^{2n\alpha x}-e^{(2n-2)\alpha x}},\quad \alpha>0,\ m=0,1,2,\ldots \text{ and } n=1,2,\ldots,\end{gathered} \tag{6.3}$$

are monotonic in x.

Remark 6.4. If one can give an answer to Open Problem 6.3, then, maybe a better upper bound for Mathieu's series (1.1) could be obtained.

Remark 6.5. In [5], several inequalities of the series $S(r,t,\alpha)$ for $0<r<1$ are established and Open Problem 6.1 is solved in the following cases:
(1) $\alpha=2$ and t is a natural number with $t>1$;
(2) $\alpha=2$ and $t=k-1/2$, where k is a natural number.

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