SOME CHAIN CONDITIONS ON WEAK INCIDENCE ALGEBRAS

SURJEET SINGH AND FAWZI AL-THUKAIR

Received 19 March 2005 and in revised form 12 July 2005

Let X be any partially ordered set, R any commutative ring, and $T = I^*(X,R)$ the weak incidence algebra of X over R. Let Z be a finite nonempty subset of X, $L(Z) = \{x \in X : x \leqslant z$ for some $z \in Z\}$, and $M = Te_Z$. Various chain conditions on M are investigated. The results so proved are used to construct some classes of right perfect rings that are not left perfect.

1. Introduction

Let R be a commutative ring and X a partially ordered set. Let $T = I^*(X,R)$ be the set of all functions $f : X \times X \to R$ such that $f(x,y) = 0$, whenever $x \nleqslant y$, and $\{(x,y) : f(x,y) \neq 0$ and $x < y\}$ is finite. Then T is an R-algebra under the operations defined as follows. For any $f,g \in T$, $x,y \in X$, and $r \in R$, $(f+g)(x,y) = f(x,y)+g(x,y)$, $fg(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$, and $rf(x,y) = r \cdot f(x,y)$. The algebra T is called *weak incidence algebra* of X over R. For a locally finite partially ordered set Y, the concept of incidence algebra $I(Y,R)$ is well known [6]. It can be proved on similar lines as for incidence algebras that for any two partially ordered sets X, Z and any two indecomposable commutative rings R, S, $I^*(X,R)$ and $I^*(Z,S)$ are isomorphic as rings if and only if X and Z are isomorphic and R and S are isomorphic [5]. It has been seen in [1, 5] that weak incidence algebras can be used to construct rings whose left and right maximal rings of quotients need not be isomorphic. Here we give some more such applications. If X is infinite, obviously T is neither left nor right artinian or Noetherian. In the present paper we study chain conditions on a specific one-sided ideal of T. Let Z be a finite nonempty subset of X, $L(Z) = \{x \in X : x \leq z$ for some $z \in Z\}$, and $M = Te_Z$, where for any subset Y of X, $e_Y \in T$ is such that $e_Y(x,x) = 1$ for every $x \in Y$, and $e_Y(x,y) = 0$ otherwise. Theorem 3.5 shows that M is an artinian left T-module if and only if R is artinian and $L(Z)$ satisfies *dcc* and has no infinite antichain. Theorem 5.2 gives a similar result for M to be Noetherian. In Section 4, the construction of partially ordered sets satisfying *dcc* but having no infinite antichains is studied. In Section 6, perfect rings are studied, as an application; Theorem 3.5 is used to construct a class of right perfect rings that are not left perfect.

International Journal of Mathematics and Mathematical Sciences 2005:15 (2005) 2389–2397
DOI: 10.1155/IJMMS.2005.2389

2. Preliminaries

Throughout, all rings have identity element $1 \neq 0$. Let X be a partially ordered set and R a commutative ring. A subset S of X is called an *antichain* in X if no two members of S are comparable [6]. We will apply the terminology for incidence algebras given in [6] for weak incidence algebras. As usual, for any $x < y$ in X, e_{xy} denotes the corresponding matrix unit in $T = I^*(X,R)$. Now $K^*(X,R) = \{f \in I^*(X,R) : f(x,x) = 0 \text{ for each } x \in X\}$ is an ideal $K^*(X,R)$ contained in its lower nil radical, and $T/K^*(X,R) \cong \Pi_{x\in X} R_x$, with each $R_x = R$. The following is immediate.

Lemma 2.1. *Let M be an artinian (Noetherian) left module over $T = I^*(X,R)$ such that $K^*(X,R)M = 0$, then for some finite subset Z of X, $(1 - e_Z)M = 0$. In particular, if M is artinian, then M has finite composition length as an R-module.*

3. Artinian modules

A partially ordered set X is said to satisfy strong *dcc* if it does not contain an infinite sequence $x_1, x_2, \dots, x_n, \dots$ such that $x_j \ngeq x_i$ whenever $i < j$. Let Z be a finite nonempty subset of X and $M = Te_Z = \sum_{x\in Z} Te_{xx}$. Now ${}_TM$ is artinian if and only if Te_{xx} is artinian for every $x \in Z$. A finite union of subsets of X satisfies strong *dcc* if and only if each of the subsets satisfies strong *dcc*. Suppose M is artinian. Then R is artinian. Suppose $L(Z)$ does not satisfy strong *dcc*. Then there exists an $x_0 \in Z$ such that $L(x_0)$ does not satisfy strong dcc. Therefore there exists an infinite sequence in $L(x_0) : x_1, x_2, \dots, x_n, \dots$ such that $x_j \ngeq x_i$ whenever $i < j$. For any $n \geq 1$, let $N_n = \sum_{k\geq n} Te_{x_k x_0}$. Then $N_{n+1} \subset N_n \subseteq M$, which contradicts the assumption that M is artinian. Hence $L(Z)$ satisfies strong *dcc*. We now discuss the converse of this result. Henceforth we assume that R is artinian and $L(Z)$ satisfies strong *dcc*. Suppose M is not artinian. Without loss of generality we take $Z = \{x_0\}$ and $M = Te_{x_0x_0}$. There exists an infinite properly descending chain of T-submodules of M : $N_1 \supset N_2 \supset \cdots \supset N_n \supset \cdots$. For each $i \geq 1$ and $x \in L(x_0)$, let $A_x^{(i)} = \{a \in R : ae_{xx_0} \in N_i\}$.

Lemma 3.1. (i) $A_x^{(i)} \subseteq A_y^{(i)}$ *whenever $y \leq x$ in $L(x_0)$.*

(ii) *For any $x \in L(x_0)$, $A_x^{(i)} \subseteq A_x^{(j)}$ whenever $j \leq i$.*

(iii) *If $A_x^{(i)} \subset A_y^{(j)}$, then either $j \leq i$ or $x \nleq y$.*

(iv) *If $A_x^{(i)} \not\subseteq A_y^{(j)}$, then either $y \nleq x$ or $i < j$.*

Proof. (i) and (ii) are obvious.

(iii) Suppose $j \nleq i$. Then $i < j$, and $A_x^{(j)} \subseteq A_x^{(i)} \subset A_y^{(j)}$. If $x \leq y$, then $A_y^{(j)} \subseteq A_x^{(j)} \subseteq A_x^{(i)}$, which is a contradiction.

(iv) Suppose $y \leq x$. Then $A_x^{(j)} \subseteq A_y^{(j)}$. If $j \leq i$, then $A_x^{(i)} \subseteq A_x^{(j)}$, therefore $A_x^{(i)} \subseteq A_y^{(j)}$, which is a contradiction. □

Let S be the set of all $A_x^{(i)}$ with $x \in L(x_0)$ and $i \geq 1$. Let $A \in S$. For some $x \in L(x_0)$ and an i, $A = A_x^{(i)}$. As $L(x_0)$ satisfies *dcc*, by keeping i fixed we can find x minimal with respect to the pair (A, i). If for some $j > i$, $A = A_x^{(j)}$, then we can find minimal $x' \leq x$ for which $A = A_{x'}^{(j)}$. Hence we can find an $x \in L(x_0)$ and a positive integer t such that $A = A_x^{(t)}$ such that if for some $u \geq t$ and $y \leq x$, $A = A_y^{(u)}$, then $x = y$. Keeping this in mind, a triple

(A,t,x) is called a *critical triple* if $A \in S$, $A = A_x^{(t)}$, and if for some $u \geq t$, $y \leq x$, $A = A_y^{(u)}$, then $y = x$. For any subset V of S, the set of those $x \in L(x_0)$ such that (A,t,x) is a critical triple for some $A \in V$ and $t \geqslant 1$ is called the $L(x_0)$-*co-support* of V.

LEMMA 3.2. (a) *Let $A,B \in S$. If for some positive integer i, (A,i,x) and (B,i,y) are critical triples and $x \neq y$, then one of the following holds:* (i) *$x < y$ and $B \subset A$,* (ii) *$y < x$ and $A \subset B$, and* (iii) *x and y are noncomparable.*

(b) *If (A,i,x) and (B,j,y) are two critical triples with A and B noncomparable or equal, and $x < y$, then $j < i$.*

Proof. (a) is immediate. (b) Now $B = A_y^{(j)} \subseteq A_x^{(j)}$. If $i \leq j$, then $A_x^{(j)} \subseteq A_x^{(i)} = A$, therefore $A = B = A_x^{(j)}$ and (A,j,y) is a critical pair. But also $A = A_x^{(j)}$, hence $x = y$, which is a contradiction. Hence $j < i$. □

LEMMA 3.3. *Let $Y \subseteq S$ be an antichain. Then Y is finite.*

Proof. Let Z be the co-support of Y. For any i, let $Y(i) = \{A \in Y : (A,i,x)$ is a critical triple for some $x \in Z\}$. Let Z_i be the set of those $x \in L(x_0)$ such that (A,i,x) is a critical triple for some $A \in Y(i)$. It follows from Lemma 3.2(a) that Z_i is an antichain, so Z_i is finite. If for some $A,B \in Y(i)$, (A,i,x) and (B,i,y) are critical triples and $A \neq B$, clearly $x \neq y$. Hence $Y(i)$ is finite. Let Z_1 be the set of minimal members of Z. Fix an $x \in Z_1$ and a critical triple (A,k,x). Consider any critical triple (B,i,y) with $x < y$ and $B \in Y$. By Lemma 3.2(b), $i < k$. Let $Y_x = \{B \in Y :$ there exists a critical triple (B,i,y) with $x \leq y\}$. It follows that $Y_x = \bigcup_{i=1}^{k}(Y_x \cap Y(i))$ is finite. As Z_1 is finite and $Y = \bigcup_{x\in Z} Y_x$, we get Y is finite. □

LEMMA 3.4. *S is finite.*

Proof. For any $A \in S$, the set S_A of all those $B \in S$ which are minimal with respect to $A < B$ is finite by Lemma 3.3. Also the set Y_1 of minimal members of S is finite. After this by using the fact that R has finite composition length, we get S is finite. □

THEOREM 3.5. *Let $T = I^*(X,R)$, where X is any partially ordered set and R is a commutative ring. Let Z be a finite nonempty subset of X and $M = Te_Z$. Then M is an artinian left T-module if and only if R is artinian and $L(Z)$ satisfies strong dcc.*

Proof. As remarked earlier it is enough to take $M = Te_{x_0x_0}$. Suppose that $L(x_0)$ satisfies strong *dcc* and R is artinian. Suppose M is not artinian. So M has an infinite properly descending chain of T-submodules: $N_1 \supset N_2 \supset \cdots \supset N_n \supset \cdots$. We use the notations given above this result. Let $A \in S$. Fix an $x \in L(x_0)$. Suppose $A = A_x^{(i)}$ for some i. Then either there exists a smallest positive integer $s_{(x,A)}$ such that $A = A_x^{(j)}$ for every $j \geq s_{(x,A)}$ or there exists a largest positive integer $k_{(x,A)}$ such that $A = A_x^{(k_{(x,A)})}$. Let Z_A be the set of those $x \in X$ for which A admits the positive integer $k_{(x,A)}$. Suppose there is no upper bound on $k_{(x,A)}$ as x ranges over Z_A. So there exists an infinite sequence: $x_1, x_2, \ldots, x_n, \ldots$ in Z_A such that $k_{(x_i,A)} > k_{(x_j,A)}$ whenever $i > j$. Then $x_i \nleq x_j$ whenever $i < j$. This contradicts the assumption that $L(x_0)$ satisfies strong *dcc*. Hence there exists a positive integer k_A such that $k_{(x,A)} < k_A$ for every $x \in Z_A$. As S is finite, we can find a positive integer u such that for any $A \in S$, $x \in L(x_0)$, $s(x,A) < u$ and $k_{(x,A)} < u$, whenever $s_{(x,A)}$ or $k_{(x,A)}$ is defined. Consider N_u. If for some $A_x^{(u)}$, $A_x^{(u)} \supset A_x^{(u+1)}$, then for $A = A_x^{(u+1)}$ we have $k_{(x,A)} > k$ or $s_{(x,A)} > u$, which is a contradiction. Hence $A_x^{(u)} = A_x^{(u+1)}$. This proves that $N_k = N_{k+1}$, which is also a contradiction. Hence M is artinian. □

Remark 3.6. Let X be a partially ordered set satisfying strong *dcc*, and R an artinian commutative ring. It follows from the above theorem that for $T = I^*(X,R)$, any finitely generated left ideal contained in $A = \sum_{x \in X} Te_{xx}$ satisfies *dcc*. As the ideal $K^*(X,R) = \{f \in T : f(x,x) = 0 \text{ for every } x \in X\} \subseteq A$, and it is nil, $K^*(X,R)$ is right T-nilpotent. However this ideal need not be left T-nilpotent. For example, let $\mathbb{N}$ be the set of natural numbers with usual ordering. Then for any field F, $K^*(\mathbb{N},F)$ is not left T-nilpotent.

4. Partially ordered sets

We now prove some results that can help in constructing partially ordered sets satisfying strong *dcc*.

Proposition 4.1. *A partially ordered set X satisfies strong dcc if and only if it satisfies dcc and it has no infinite antichain.*

Proof. If X satisfies strong *dcc*, obviously it cannot have an infinite antichain. Conversely, let X satisfy strong *dcc* and have no infinite antichain. Suppose there exists an infinite sequence $\{x_i\}$ in X such that $x_j \not\geq x_i$ whenever $i < j$. These x_i are distinct. Let A be the set of these x_i and S the set of minimal members of A. Then S is a finite nonempty set. So there exists an $x_i \in S$ such that $x_i < x_j$ for infinitely many values of j. As a consequence, we can find a $k > i$ such that $x_i < x_k$, which is a contradiction. Hence X satisfies strong *dcc*. □

Theorem 4.2. *Let X and Y be two partially ordered sets satisfying strong dcc, then the partially ordered set $Z = X \times Y$ with the ordering given by $(a,b) \leq (c,d)$ if and only if $a \leq c$ and $b \leq d$ satisfies strong dcc.*

Proof. That Z satisfies *dcc* is obvious. Suppose Z has an infinite antichain S. Let A_1 and A_2 be sets of X-components and Y-components respectively of the members of S. As Y does not contain an infinite antichain, for any fixed $x \in A_1$, there are only finitely many $y \in A_2$ such that $(x,y) \in S$. Also, the number of minimal members of A_1 is finite. So there exists a minimal member $x_1 \in A_1$ such that $T_1 = \{(x,y) \in S : x_1 < x\}$ is infinite. Fix an $(x_1,y_1) \in S$. If $(x,y) \in T_1$, then either $y < y_1$ or y and y_1 are noncomparable. Thus T_1 satisfies one of the following conditions.

(i) There are infinitely many $(x,y) \in T_1$ such that $y < y_1$.

(ii) There are infinitely many $(x,y) \in T_1$ such that y and y_1 are noncomparable.

Suppose (i) is satisfied. Then $S_1 = \{(x,y) \in T_1 : y < y_1\}$ is infinite. As for S, we can find an $(x_2,y_2) \in S_1$ such that $T_2 = \{(x,y) \in S_1 : x_2 < x\}$ is infinite. Now $y_2 < y_1$. Suppose T_2 also satisfies (i), that gives rise to a subset S_2 analogous to S_1. Continue the process, and this gives a descending chain in Y. As Y satisfies *dcc*, this process must end after a finite number of steps. Thus we get a subset V_1 of S_1 and an element $(u_1,v_1) \in V_1$ such that $V_2 = \{(x,y) \in V : u_1 < x,\ y \text{ and } v_1 \text{ are not comparable}\}$ is infinite. Thus for any infinite antichain S in Z, there exists a $(u,v) \in S$, such that $T = \{(x,y) \in S : u < x,\ y \text{ is not comparable with } v\}$ is infinite, so T satisfies (ii). Suppose for some $n \geq 2$, we have constructed infinite sets V_i in S, for $1 \leq i \leq n$, $(u_i, v_i) \in V_i$, for $1 \leq i \leq n-1$ with $V_{i+1} = \{(x,y) \in V_i : u_i < x,\ v_i \text{ and } y \text{ are noncomparable}\}$. Now V_n has an element (u_n, v_n) such that $V_{n+1} = \{(x,y) \in V_n : u_n < x,\ v_n \text{ and } y \text{ are noncomparable}\}$ is infinite. This inductive process gives an infinite set $L = \{(u_i, v_i) : i \geq 1\} \subseteq S$ such that $u_i < u_{i+1}$ for any $i \geq 1$, but $B = \{v_i : i \geq 1\}$ is an infinite antichain in Y. This is a contradiction. Hence Z satisfies strong *dcc*. □

Example 4.3. For any finite collection of well-ordered sets, their direct product as defined in the above theorem satisfies strong *dcc*.

Definition 4.4. Let X be a partially ordered set satisfying *dcc*. For any nonnegative integer, define $s_i(X)$ as follows. Firstly, $s_0(X)$ is the set of all minimal elements in X. For any $i \geq 0$, an $x \in s_{i+1}(X)$, if it is minimal with respect to the property that for some $y \in s_i(X)$, $y < x$. Define $B_1(X) = \bigcup_{i \geq 0} s_i(X)$.

Lemma 4.5. *Let X be any partially ordered set satisfying dcc.*

(i) *Every $s_i(X)$ is an antichain. In addition if X satisfies strong dcc, then every $s_i(X)$ is finite.*

(ii) *If for some $i > 0$, an $x \in s_i(X)$, then there exists a sequence $x_0 < x_1 < \cdots < x_i = x$ such that $x_j \in s_j(X)$ for $0 \leq j \leq i$.*

(iii) *Let $x \in s_i(X)$ for some i, $y \in s_j(X)$ for some $j > i$. Then $y \nleq x$.*

Proof. (i) is immediate from the definition and Proposition 4.1.

(ii) follows by using induction on i.

(iii) Suppose $y \leq x$. By using (ii) we have $y_{i-1} < y \leq x$. By Definition 4.4, $y = x$. At the same time, as $j > i$, by (ii), there exists $z \in s_i(X)$ such that $z < y$. This contradicts (i). Hence the result follows. □

Definition 4.6. Let X be a partially ordered set satisfying *dcc*. For any ordinal α, define $B_\alpha(X)$ as follows. $B_0(X) = \emptyset$, the empty set, if $\alpha = \beta + 1$, then $B_\alpha(X) = B_\beta(X) \cup B_1(X \backslash B_\beta(X))$. If α is a limit ordinal, then $B_\alpha(X) = \bigcup_{\beta < \alpha} B_\beta(X)$.

Lemma 4.7. *Let X be any partially ordered set satisfying strong dcc.*

(i) *$B_1(X)$ is countable.*

(ii) *For any two ordinals $\beta < \alpha$, if $\alpha = \beta + \gamma$, then $B_\alpha(X) = B_\beta(X) \cup B_\gamma(X \backslash B_\beta(X))$.*

(iii) *$X = B_\alpha(X)$ for some ordinal α.*

(iv) *Suppose $X = B_\alpha(X)$ for some smallest ordinal α. If for every $\beta < \alpha$, $B_1(X \backslash B_\beta(X))$ is linearly ordered, then X is linearly ordered.*

Proof. (i) is immediate from Lemma 4.5.

(ii) follows from Definition 4.4 by using transfinite induction on γ.

(iii) If $X = B_1(X)$, there is nothing to prove. Suppose $X \neq B_1(X)$. Then $B_1(X)$ is countably infinite. It follows from the definition of a $B_\beta(X)$ that if $X \neq B_\beta(X)$, then $|B_\beta(X)| \geq |\beta|$. Now there exists a smallest ordinal β such that $|\beta| > |X|$. Then $X = B_\beta(X)$. Finally (iv) is obvious. □

Remark 4.8. Let X be a partially ordered set satisfying strong *dcc*. If X is infinite, then each $s_i(X)$ is nonempty and $B_1(X)$ is countably infinite. So, the given ordering on $B_1(X)$ can be extended to a linear ordering such that $B_1(X)$ becomes isomorphic to the set of natural numbers. Now extend the ordering on X as follows. Let $x, y \in X$. If $x \in B_\alpha(X)$ and $y \in B_\beta(X)$ such that $\alpha < \beta$ and $y \notin B_\alpha(X)$, then set $x < y$. For any ordinal α, extend the ordering on $B_1(X)\backslash B_\alpha(X)$, such that it embeds in the set of natural numbers. This makes X a linearly ordered set satisfying *dcc*. The order on any partially ordered set can be extended to a linear order, [3, Chapter 1]. Here, we see that X can be made into a well-ordered set. Let $(Y, \leqslant)$ be any linearly ordered set with the following properties. (i) For some ordinal α, Y is a union of an ascending chain of subsets $\{Y_\beta\}_{\beta\le\alpha}$, with each $Y_{\beta+1}\backslash Y_\beta$ embeddable in the set of natural numbers. (ii) For any limit ordinal $\beta \le \alpha$, $Y_\beta = \bigcup_{\gamma<\beta} Y\gamma$. (iii) For any $y \in Y\backslash Y_\beta$ and $x \in Y_\beta$, $x < y$. Then Y satisfies *dcc*. For each $\beta < \alpha$, consider any ordering $\leqslant_\beta$ on $Y_{\beta+1}\backslash Y_\beta$ under which $Y_{\beta+1}\backslash Y_\beta$ satisfies strong *dcc* and the given ordering on $Y_{\beta+1}\backslash Y_\beta$ extends $\leqslant_\beta$. This defines an ordering $\leqslant'$ on Y that for each $\beta < \alpha$ coincides with $\leqslant_\beta$ on $Y_{\beta+1}\backslash Y_\beta$ and equals $\leqslant$ otherwise. Then $(Y, \leqslant')$ satisfies strong *dcc*.

5. Noetherian modules

Let X be a partially ordered set. X is said to satisfy strong *acc* if it does not contain an infinite sequence $x_1, x_2, \dots, x_n, \dots$ such that $x_j \nleq x_i$ whenever $j > i$.

As in Section 3, we consider $M = Te_Z$, where Z is a finite nonempty subset of X. If ${}_TM$ is Noetherian, it follows on similar lines as in Section 3 that R is Noetherian and $L(Z)$ satisfies strong *acc*.

To prove the converse of the above remark, throughout we take R to be Noetherian, $Z = \{x_0\}$, and $x_0 \in X$ such that $L(x_0)$ satisfies strong *acc*. Let N be a submodule of M. For each $x \in L(x_0)$, set $A_x = \{a \in R : ae_{xx_0} \in N\}$. Each A_x is an ideal of R and $N = \sum_{x\in L(x_0)} A_x e_{xx_0}$. For $x \le y$ in $L(x_0)$, $A_y \subseteq A_x$. Let S be the set of all A_x, $x \in L(x_0)$. Consider any subset K of S. For any $A \in K$, as $L(x_0)$ satisfies *acc*, we can find $x \in L(x_0)$ maximal with respect to the property that $A = A_x$. Let $Z(K)$ be the set of all such maximal elements of $L(x_0)$.

Lemma 5.1. *Let $Y \subseteq S$ be an antichain. Then $Z(Y)$ is an antichain and Y is finite.*

Proof. Let $x, y \in Z(Y)$ such that $x \le y$. For some $A, B \in Y$, $A = A_x$ and $B = A_y$. However $A_y \subseteq A_x$, so $A = B$. As x is maximal with respect to A, we get $x = y$. Hence $Z(Y)$ is an antichain, so $Z(Y)$ is finite. For each $A \in Y$, there exists an $x \in Z(Y)$ such that $A = A_x$. Thus there exists a mapping of $Z(Y)$ onto Y. Hence Y is finite. □

Theorem 5.2. *Let $T = I^*(X,R)$ where X is a partially ordered set and Z is a finite nonempty subset of X. Then $M = Te_Z$ is a Noetherian T-module if and only if R is Noetherian and $L(x_0)$ satisfies strong acc.*

Proof. Without loss of generality we take $Z = \{x_0\}$. We use notations given above Lemma 5.1. Let R be Noetherian and $L(x_0)$ satisfy strong *acc*. Let N be a T-submodule of M. As R is Noetherian, $Y_1 = \{A \in S : A \text{ is maximal in } S\}$ is nonempty and no two members of Y_1 are comparable. Set $Z_1 = Z(Y_1)$. Consider $N_1 = \sum_{x\in Z_1} TA_x e_{xx_0}$. Let $y \le x$ with $x \in Z_1$, then $A_x \subseteq A_y$, therefore $A_y = A_x$ and $A_y e_{yx_0} = e_{yx}(A_x e_{xx_0}) \subseteq N_1$. Hence $N_1 = \sum_{x\in L(Z_1)} A_x e_{xx_0}$. Suppose, for some $n \ge 1$, we have already defined subsets $Z_1, Z_2, \dots, Z_n$, $V_n = \bigcup_{i=1}^n Z_i$, and

$N_n = \sum_{x\in V_n} TA_x e_{xx_0}$ such that the following hold. (i) $N_n = \sum_{x\in L(V_n)} A_x e_{xx_0}$, (ii) for any $y \in L(V_n)$, there exists an x in V_n such that $y \leq x$ and $A_y = A_{x_i}$, and (iii) for any $y \in L(x_0)\backslash L(V_n)$, there exists $x \in Z_n$ such that $A_y < A_x$. Set $S_{n+1} = \{A \in S : A = A_x$ for some $x \in L(x_0)\backslash L(V_n)\}$ and Y_{n+1} the set of all maximal members of S_{n+1}. Set $Z_{n+1} = Z(Y_{n+1})$, $V_{n+1} = V_n \cup Z_{n+1}$, and $N_{n+1} = \sum_{x\in V_n} TA_x e_{xx_0}$. The above three conditions are obviously satisfied by N_1. Suppose they are satisfied by N_n for some n. Suppose $y \in Z_{n+1}$ and $x \in X$ such that $x < y$. Then $A_y \subseteq A_x$. If $x \notin L(V_n)$, $A_x = A_y$. If $y \in L(V_n)$, by (ii) there exists $z \in V_n \subseteq V_{n+1}$ such that $y \leq z$ and $A_y = A_z$. Hence $N_{n+1} = \sum_{x\in L(V_{n+1})} A_x e_{xx_0}$. Thus N_{n+1} satisfies (i), (ii), and (iii). For each i for which Z_i is non-empty, fix an $x_i \in Z_i$. If an $L(Z_i) \neq$ S, obviously $Z_{i+1} \neq \emptyset$. For $i < j$, as $L(V_i) \cap Z_j = \emptyset$, $x_j \nleq x_i$. As $L(x_0)$ satisfies strong *acc*, it follows that there exists an n such that $Z_n \neq \emptyset$ but $Z_{n+1} = \emptyset$. Consequently, $L(x_0) = L(V_n)$, $N_n = N$. As V_n is finite, each A_x is finitely generated as an R-module, and $N = \sum_{x\in V_n} TA_x e_{xx_0}$, it follows that N is a finitely generated T-module. Hence M is Noetherian. □

Remark 5.3. Let X' be the dual of a partially ordered set X. For any commutative ring R, set $T' = I^*(X', R)$ and $T = I^*(X, R)$. These two algebras are naturally anti-isomorphic. Let Z be a finite nonempty subset of X, $M = e_Z T$, and $U(Z) = \{x \in X : x \geq z$ for some $z \in Z\}$. By using the anti-isomorphism between T and T' and Theorems 3.5 and 5.2, we get the following results:

(i) M_T is artinian if and only if R is artinian and $U(Z)$ satisfies strong *acc*;
(ii) M_T is Noetherian if and only if R is Noetherian and $U(Z)$ satisfies strong *dcc*.

Remark 5.4. Let X be a locally finite partially ordered set, and $T = I(X, R)$ the incidence algebra of X over a commutative ring R. Suppose R is artinian and for some $x_0 \in X$, $L(x_0)$ satisfies strong *dcc*. As $L(x_0)$ has finitely many minimal elements, $L(x_0)$ is a finite set, so $M = Te_{x_0x_0}$, being a finite direct sum of copies of R, is trivially an artininian left T-module. Hence M is an artinian left T-module if and only if R is artinian and $L(x_0)$ satisfies strong *dcc*.

Now suppose R is Noetherian, $L(x_0)$ satisfies strong *acc*, and N is a T-submodule of M. As in the proof of Theorem 5.2, we have Y_1 and $Z_1 = Z(Y_1)$. Consider N_n as defined in the proof of Theorem 5.2. Now $N_1 = \sum_{x\in Z_1} TA_x e_{xx_0}$. For any $x \in Z_1$, let $G_x = \{b_{xj} : 1 \leqslant j \leqslant n_x\}$ generate A_x as an R-module. Consider any $f \in N_1$ with $D_f = \{z \in L(x_0) : f(z, x_0) \neq 0\} \subseteq L(Z_1)$. Let $z \in L(Z_1)$. Then for any $x \in Z_1$, $A_z = A_x$ whenever $z \leqslant x$. So $f(z, x_0) = \sum_{z\leqslant x} \sum_{j=1}^{n_x} r_{zxj} b_{xj}$, where $x \in Z_1$. Then the formal sum $g_{xj} = \sum_{z\leqslant x} r_{zxj} e_{zx} \in T$ and $f = \sum_{x\in Z_1} \sum_{j=1}^{n_x} g_{xj} b_{xj} e_{xx_0} \in N_1$. Hence $N_1 = \{f \in N_1 : D_f \subseteq L(Z_1)\}$. Inductively, one can prove that for any $n \geqslant 1$, $N_n = \{f \in N : D_f \subseteq L(V_n)\}$. Each N_n is finitely generated. Hence as in Theorem 5.2, we get that $N = N_n$ for some n, hence N is finitely generated. This proves that M is a Noetherian left T-module if and only if R is Noetherian and $L(x_0)$ satisfies strong *acc*.

6. Perfect rings

A partially ordered set X is said to locally satisfy *strong dcc*, if for any finite subset S of X, $L(S)$ satisfies strong *dcc*. Throughout, R is an artinian, commutative local ring, X is a partially ordered set locally satisfying strong *dcc*, and $T = I^*(X,R)$. Let $T' = R + K^*(X,R)$. Then T' is a local ring. We will prove that T' is right perfect. We will write K for $K^*(X,R)$.

Lemma 6.1. *Any finitely generated left ideal of T' contained in K^* is artinian.*

Proof. Let ${}_TC$ be any artinian module. By Lemma 2.1, C/K^*C is of finite composition length over R. Let $A = \sum_{i=1}^{n} T'b_i$ be a finitely generated left ideal of T' contained in K^*. Then $B = \sum_{i=1}^{n} Tb_i$ is a finitely generated left ideal of T contained in K^*. Let $A = A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending chain of left ideals of T'. As ${}_TB$ is artinian, there exists a positive integer m such that $K^*A_i = K^*A_m$ for any $i \geq m$. Let $B_i = TA_i$. Then $K^*B_i = K^*A_i$. Now B_i is an artinian left T-module. It follows that for any $i \geq m$, B_i/K^*B_i is of finite composition length over R. Therefore there exists an $n \geq m$ such that $A_j/K^*A_j = A_j/K^*A_n = A_n/K^*A_n$ for any $j \geq n$. Hence $A_j = A_n$ for any $j \geq n$. □

Theorem 6.2. *T' is a local, right perfect ring.*

Proof. It is enough to prove that T' satisfies *dcc* on principal left ideals [2, Theorem 28.4]. Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be a descending chain of principal left ideals of T'. In view of Lemma 6.1, we take $A_i = T'(\alpha_i I + b_i)$ for some $\alpha_i \neq 0$ in R and $b_i \in K^*$. Then $\alpha_{i+1}I + b_{i+1} = (\beta_i I + c_i)(\alpha_i I + b_i)$ for some $\beta_i \in R$, and $c_i \in K^*$. This gives $\alpha_{i+1} = \beta_i\alpha_i$ and $ann_R(\alpha_i) \subseteq ann_R(\alpha_{i+1})$. As R is Noetherian, there exists a positive integer m such that $ann_R(\alpha_i) = ann_R(\alpha_{i+1})$ for any $i \geq m$. Therefore β_i is a unit for any $i \geq m$ and $\beta_i I + c_i$ is a unit. Hence $A_i = A_m$ for any $i \geq m$. □

The dualization of the above result gives the following.

Theorem 6.3. *Let X be a partially ordered set such that for any finite nonempty subset Z of X, $U(Z)$ satisfies strong acc, R is an artinian commutative ring, and $T = I^*(X,R)$. Then $T' = R + K^*(X,R)$ is left perfect.*

Examples of rings that are right perfect but not left perfect are well known (one such example is the dual of example given in [2, Exercise 2, page 322]). By using the above theorem, we end this section by constructing a class of right perfect rings that are not left perfect.

Example 6.4. Let X be any partially ordered set that locally satisfies strong dcc, but has a finite, nonempty subset Z such that $L(Z)$ is not finite. As $L(Z)$ satisfies strong *dcc*, $L(Z)$ has a subset V isomorphic to the set of natural number. Any infinite well-ordered set not embeddable in the set of natural numbers is such a set X. Thus V is given by elements: $x_1 < x_2 < \cdots < x_n < \cdots$. Let R be a local artinian ring, and $T' = R + K^*(X,R)$. By Theorem 6.2, T' is right perfect, however $\{e_{x_1x_i}T'\}_{i\geq 2}$ is an infinite, nonterminating descending sequence of principal right ideals in T'. Hence T' is not left perfect.

Acknowledgment

The authors are thankful to the referees for their comments and suggestions.

References

[1] F. Al-Thukair, S. Singh, and I. Zaguia, *Maximal ring of quotients of an incidence algebra*, Arch. Math. (Basel) **80** (2003), no. 4, 358–362.

[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, vol. 13, Springer, New York, 1974.

[3] P. C. Fishburn, *Interval Orders and Interval Graphs*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons, New York, 1985.

[4] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, vol. 189, Springer, New York, 1999.

[5] S. Singh and F. Al-Thukair, *Weak incidence algebra and maximal ring of quotients*, Int. J. Math. Math. Sci. **2004** (2004), no. 53, 2835–2845.

[6] E. Spiegel and C. J. O'Donnell, *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 206, Marcel Dekker, New York, 1997.

Surjeet Singh: Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh 11451, Kingdom of Saudi Arabia

E-mail address: ssingh@ksu.edu.sa

Fawzi Al-Thukair: Department of Mathematics, King Saud University, P.O. Box 2455, Riyadh 11451, Kingdom of Saudi Arabia

E-mail address: thukair@ksu.edu.sa