

ON A PROBLEM ON EXPLICIT EMBEDDINGS OF THE GROUP \mathbb{Q}

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To Professor Alfred L. Shmelkin on his birthday

Answering a question of de la Harpe and Bridson in the Kourovka Notebook, we build the explicit embeddings of the additive group of rational numbers \mathbb{Q} in a finitely generated group G . The group G in fact is two-generator, and the constructed embedding can be subnormal and preserve a few properties such as solubility or torsion freeness.

1. Introduction

In 1999, de la Harpe and Bridson grouped a few “well-known” questions as Problem 14.10 in Kourovka notebook [7]. These questions mainly concern *explicit* embeddings of some (finitely or infinitely presented) countable groups into finitely generated groups. In particular, they posed the following possibly easier problem.

PROBLEM 1.1 [7, from Problem 14.10]. *Find an explicit embedding of the additive group of rational numbers \mathbb{Q} in a finitely generated group; such a group exists by [4, Theorem IV].*

An explicit embedding mentioned in this problem is possible and, moreover, the finitely generated group G can in fact be two-generator. This, of course, stresses the similarity of the constructed embedding with the embedding of [4] because the latter, too, is an embedding into a two-generator group.

In the current paper, we suggest two mechanisms to build the embedding mentioned. The first argument uses the operation of wreath products, while the second argument is based on Higman-Neumann-Neumann extensions (HNN extensions) of groups.

For general group-theoretical background information, we refer to [5, 15]. More specific information on wreath products can be found in [12], and on generalized free products and HNN extensions can be found in [6, 14]. For original methods and techniques of work with embeddings of countable groups in two-generator groups, we refer to well-known articles [2, 3, 4, 11].

2. An embedding based on wreath products

In our first construction, the group G will be built as a two-generator subgroup in the nested cartesian wreath product $W = (\mathbb{Q} \text{Wr } C) \text{Wr } Z$, where $C = \langle c \rangle$ and $Z = \langle z \rangle$ are

infinite cyclic groups (written multiplicatively). Firstly, for each positive integer n , choose in the base subgroup \mathbb{Q}^C of the cartesian wreath product $\mathbb{Q} \text{Wr} C$ the elements φ_n and ψ_n :

$$\varphi_n(c^i) = \begin{cases} \frac{1}{n} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0, \end{cases} \quad \psi_n(c^i) = \begin{cases} \frac{1}{n} & \text{if } i < 0, \\ 0 & \text{if } i \geq 0. \end{cases} \quad (2.1)$$

The reason of such selection is in the following relations:

$$[\psi_n, c] = \varphi_n, \quad [\psi_n, \psi_k] = 1 \quad \text{for any } n, k > 0. \quad (2.2)$$

The first of the relations (2.2) follows from

$$[\psi_n, c](c^i) = \psi_n^{-1} \psi_n^c(c^i) = \begin{cases} -\frac{1}{n} + \frac{1}{n} = 0 = \varphi_n(c^i) & \text{if } i < 0, \\ 0 + \frac{1}{n} = \frac{1}{n} = \varphi_n(c^0) & \text{if } i = 0, \\ -0 + 0 = 0 = \varphi_n(c^i) & \text{if } i > 0. \end{cases} \quad (2.3)$$

The second of the relations (2.2) trivially follows from the fact that \mathbb{Q}^C is abelian.

In the base subgroup $(\mathbb{Q} \text{Wr} C)^Z$ of W , take an element α defined as

$$\alpha(z^j) = \begin{cases} 1 = 1_{\mathbb{Q} \text{Wr} C} & \text{if } j < 0, \\ c & \text{if } j = 0, \\ \psi_j & \text{if } j > 0. \end{cases} \quad (2.4)$$

Put $G = \langle \alpha, z \rangle$ and define the embedding $\Phi : \mathbb{Q} \rightarrow G$ as

$$\Phi : \frac{m}{n} \mapsto [z^n \alpha z^{-n}, \alpha]^m = [\alpha^{z^{-n}}, \alpha]^m \quad \text{for any } \frac{m}{n} \in \mathbb{Q}, n > 0. \quad (2.5)$$

That Φ is a homomorphism and an injection could be checked directly. But to avoid very long calculations, we consider the structure of the commutator $[\alpha^{z^{-n}}, \alpha]$ first:

$$[\alpha^{z^{-n}}, \alpha](z^j) = [\alpha(z^{j+n}), \alpha(z^j)] = \begin{cases} [1, 1] = 1 & \text{if } j < -n, \\ [c, 1] = 1 & \text{if } j = -n, \\ [\psi_{j+n}, 1] = 1 & \text{if } -n < j < 0, \\ [\psi_n, c] = \varphi_n & \text{if } j = 0, \\ [\psi_{j+n}, \psi_j] = 1 & \text{if } j > 0 \end{cases} \quad (2.6)$$

(the last two lines follow from (2.2)). This means that $[\alpha^{z^{-n}}, \alpha]$ is nothing else but the image φ_n^* of the coordinate element φ_n in the “the first copy” of the group $\mathbb{Q} \text{Wr} C$ in W :

$$\varphi_n^*(z^j) = \begin{cases} \varphi_n & \text{if } j = 0, \\ 1 = 1_{\mathbb{Q} \text{Wr} C} & \text{if } j \neq 0. \end{cases} \quad (2.7)$$

Therefore the elements

$$\Phi(\mathbb{Q}) = \left\{ \Phi\left(\frac{m}{n}\right) = (\varphi_n^*)^m \mid \frac{m}{n} \in \mathbb{Q}, n > 0 \right\} \tag{2.8}$$

do form a subgroup isomorphic to \mathbb{Q} in G , and the mapping Φ is injective. Finally, it easily follows from the equalities

$$\Phi\left(\frac{1}{n}\right) \cdot \Phi\left(\frac{1}{n'}\right) = \varphi_n^* \cdot \varphi_{n'}^* = \varphi_{nn'}^*, \quad \Phi\left(\frac{m}{n}\right) = (\varphi_n^*)^m = \underbrace{\varphi_n^* + \dots + \varphi_n^*}_{m \text{ times}} \tag{2.9}$$

that Φ is a homomorphism (where in the formula above “+” is used in the sense that all the coordinates of the summands are added as rational numbers).

3. An embedding based on HNN extension

This time, the group G will be built as a certain HNN extension of the free product $H = \mathbb{Q} * F_2$, where $F_2 = \langle x, y \rangle$ is a free group of rank 2. For each positive integer n in the group H , choose a pair of elements

$$u_n = x^{(y^{n-1})}, \quad v_n = y^{(x^{n-1})} \cdot \frac{1}{n}. \tag{3.1}$$

Evidently, for any $n > 0$, the words u_n and v_n are reduced in H and, in particular, are nontrivial. Therefore, the map $f' : u_n \mapsto v_n, n > 0$ can be continued to an isomorphism f of subgroups

$$U = \langle u_n | n > 0 \rangle, \quad V = \langle v_n | n > 0 \rangle \tag{3.2}$$

of H . Take an “external” stable letter t and build the HNN extension $N = \langle H, t \rangle$ defined by

$$u_n^t = f(u_n) = v_n \quad \forall n > 0. \tag{3.3}$$

Put $G = \langle x, t \rangle \leq N$ and define the embedding $\Psi : \mathbb{Q} \rightarrow G$ as

$$\Psi : \frac{m}{n} \mapsto \left(\left((x^t)^{-1} \right)^{x^{n-1}} \left(x^{(x^t)^{n-1}} \right)^t \right)^m \quad \text{for any } \frac{m}{n} \in \mathbb{Q}, n > 0. \tag{3.4}$$

Here, too, it can be checked directly that this is a homomorphism and is an injection. But, again, to avoid long calculations, we consider the above components of $\Psi(m/n)$.

Since $x^t = u_1^t = f(u_1) = v_1 = y \cdot 1/1 = y$, the group G contains y . Therefore, for each $n > 0$,

$$\left((x^t)^{-1} \right)^{x^{n-1}} = (y^{-1})^{x^{n-1}} = (y^{x^{n-1}})^{-1} \tag{3.5}$$

holds. On the other hand,

$$\left(x^{(x^t)^{n-1}} \right)^t = \left(x^{(y^{n-1})} \right)^t = f(u_n) = v_n = y^{(x^{n-1})} \cdot \frac{1}{n}. \tag{3.6}$$

The product of (3.5) and (3.6) is then equal to $1/n$ (in fact, to the copy of $1/n$ inside the HNN extension of the free product $\mathbb{Q} * F_2$).

Therefore the elements

$$\Psi(\mathbb{Q}) = \left\{ \Psi\left(\frac{m}{n}\right) \mid \frac{m}{n} \in \mathbb{Q}, n > 0 \right\} \tag{3.7}$$

do form a subgroup isomorphic to \mathbb{Q} . The map Ψ is injective, and evidently is a homomorphism. We may also notice that, in fact, $G = N$ because N contains a set generating G , namely,

$$\left\{ x, y = x^t, \frac{1}{n} = \Psi\left(\frac{1}{n}\right) \mid n > 0 \right\}. \tag{3.8}$$

4. Some comparison and additional properties for embeddings

Of these two embeddings Φ and Ψ , the second is much closer to the original embedding of Higman et al. [4]. However, the embedding Φ is much more economical, because the abelian group \mathbb{Q} is embedded into a soluble group of length 3. Whereas Ψ embeds \mathbb{Q} in a group that contains a subgroup isomorphic to the free group of infinite rank (and, therefore, generates the variety of all groups).

On the other hand, for the embedding Φ , we needed the fact that \mathbb{Q} is abelian; it was required for the equality $[\psi_n, \psi_k] = 1$ in (2.2). This means that the concept of Φ can only be used to build explicit embedding of countable *abelian* groups given by their generators. Whereas the concept of Ψ is suitable for nonabelian groups, as well.

However, there are a few more complex versions of the embedding Φ that allow to also build explicit embeddings of nonabelian groups. But here we restrict our consideration to this simpler form of Φ and refer to [10] for further variations of the argument.

Another addition to the above construction of Φ is that the group G can be modified to have a *full order* (in the sense of [13]) that continues the “normal” order in \mathbb{Q} . That is, G can be built to have an order relation “ $<$ ” such that

- (i) “ $<$ ” is a linear order on G ;
- (ii) if $m/n < m'/n'$ in \mathbb{Q} , then also $\Phi(m/n) < \Phi(m'/n')$ in G ;
- (iii) for any $g, h, t \in G$, if $g < h$, then also $gt < ht$ and $tg < th$.

For further development of this subject, see our recent work [8, 9, 10].

The embeddings Φ and Ψ both preserve the torsion freeness of \mathbb{Q} . The two-generator group built for Φ is torsion free because it is a subgroup of the torsion-free group W [12].

That the two-generator group built for Ψ is torsion free follows from the following. Since \mathbb{Q} and F_2 both are torsion free, the group $H = \mathbb{Q} * F_2$ evidently is torsion free. By [6, Theorem 2.4], if the group $G = N$ has an element g of finite order, then it is conjugate with an element of the base subgroup H . Since the latter is torsion free, $g = 1$ holds.

Finally, we notice that the embedding Φ is subnormal: $\Phi(\mathbb{Q}) \triangleleft \triangleleft G$, because the first copy of \mathbb{Q} is normal in \mathbb{Q}^C . The latter is normal in $\mathbb{Q} \text{Wr} C$. The first copy of $\mathbb{Q} \text{Wr} C$ is normal in $(\mathbb{Q} \text{Wr} C)^Z$, and this base subgroup is normal in W . Subnormality of embeddings of countable groups into two-generator groups was first proved by Dark in [1].

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