EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SEMILINEAR ELLIPTIC SYSTEM

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Received 23 February 2005 and in revised form 4 May 2005

We consider the existence, the nonexistence, and the uniqueness of solutions of some special systems of nonlinear elliptic equations with boundary conditions. In a particular case, the system reduces to the homogeneous Dirichlet problem for the biharmonic equation $\Delta^2 u = |u|^p$ in a ball with $p > 0$.

1. Introduction

In this paper, we are interested in the existence, the nonexistence, and the uniqueness question for the following problem:

$$\begin{aligned} \Delta u &= |v|^{q-1}v \quad \text{in } B_R, \\ \Delta v &= |u|^p \quad \text{in } B_R, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R, \end{aligned} \tag{1.1}$$

where B_R denotes the open ball of radius R centered at the origin in $\mathbb{R}^n$ $(n \geq 1)$, $\partial/\partial\nu$ is the outward normal derivative, and $p, q > 0$.

Concerning uniqueness, we have the following theorem.

Theorem 1.1. (i) *Let $p > 0$, $q \geq 1$ with $pq \neq 1$. Then (1.1) has at most one nontrivial radial solution $(u,v) \in (C^2(\overline{B}_R))^2$.*

(ii) *Let $p > 0$, $q \geq 1$ with $pq = 1$. Assume that (1.1) has a nontrivial radial solution $(u,v) \in (C^2(\overline{B}_R))^2$. Then all nontrivial radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.*

When $q = 1$ and $p \in (0,1) \cup (1,\infty)$, Theorem 1.1 was established in [4] (see also the references therein). When $n = 1$, $q = 1$, and $p > 1$, the uniqueness of a nontrivial solution follows from a general result given in [5].

When $q = 1$, $p > 1$, and

$$p < \frac{n+4}{n-4} \quad \text{if } n \geq 5, \tag{1.2}$$

International Journal of Mathematics and Mathematical Sciences 2005:10 (2005) 1507–1523
DOI: 10.1155/IJMMS.2005.1507

the existence of a nontrivial solution was proved in [2, 5, 11]. The case $q = 1$ and $0 < p < 1$ is well known: see, for instance, [4, 6]. Moreover, when $q = 1$, any nontrivial solution of (1.1) is positive in B_R because the Green function of Δ^2 with Dirichlet boundary conditions is positive in B_R [1, 8]. Then it was proved in [2, 11, 12] that problem (1.1) has no nontrivial solutions, whether radial or not, if

$$p \geq \frac{n+4}{n-4} \quad (n \geq 5). \tag{1.3}$$

We will prove a nonexistence result and an existence result.

Theorem 1.2. *Suppose $n \geq 3$. Let $p, q > 0$ satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}. \tag{1.4}$$

(i) *Let $(u,v) \in (C^2(\overline{B}_R))^2$ be a solution of problem (1.1) such that $u \geq 0$ in B_R. Then $u = v = 0$.*

(ii) *If $(u,v) \in (C^2(\overline{B}_R))^2$ is a radial solution of problem (1.1), then $u = v = 0$.*

Theorem 1.3. (i) *Let $p > 0$, $q \geq 1$ with $pq \neq 1$ satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n} \quad \text{if } n \geq 3. \tag{1.5}$$

Then (1.1) has a nontrivial radial solution $(u,v) \in (C^2(\overline{B}_R))^2$.

(ii) *Let $p > 0$, $q \geq 1$ with $pq = 1$. Then there exists $R > 0$ such that (1.1) has a nontrivial radial solution $(u,v) \in (C^2(\overline{B}_R))^2$.*

Remark 1.4. Notice that when $pq \leq 1$, (1.5) holds.

In the sequel, Δ denotes equally the Cartesian and the polar form of the Laplacian.

In Section 2, we give some preliminary results. Theorem 1.1 is proved in Section 3 using the same approach as in [4, 7]. In Section 4, we prove Theorem 1.2. We prove Theorem 1.3 in Section 5: the proof is based on a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. The fact that $q \geq 1$ is crucial in the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

In this section, we first examine the structure of nontrivial radial solutions of (1.1).

Lemma 2.1. *Let $(u,v) \in (C^2(\overline{B}_R))^2$ be a nontrivial radial solution of (1.1). Then $u' < 0$ on $(0,R)$, $\Delta u(R) = u''(R) > 0$ and $v' > 0$ on $(0,R]$, $v(0) < 0 < v(R)$.*

Proof. Clearly $u = 0$ if and only if $v = 0$. We have

$$r^{n-1}v'(r) = \int_0^r s^{n-1} |u(s)|^p ds \geq 0, \quad 0 \leq r \leq R. \tag{2.1}$$

Assume that $v(0) \geq 0$. Then (2.1) implies that $v \geq 0$ on $[0,R]$, hence $\Delta u \geq 0$ on $[0,R]$. Therefore $r^{n-1}u'(r)$ is nondecreasing in $[0,R]$. Since $u'(0) = u'(R) = u(R) = 0$, we deduce that $u = 0$ and we reach a contradiction. The case where $v(R) \le 0$ can be handled in the same way. Therefore we have $v(0) < 0 < v(R)$. We claim that $u(0) \ne 0$. Indeed assume that $u(0) = 0$. Using (2.1) and the first equation in (1.1), we deduce that there exists $R' \in (0,R)$ such that $r^{n-1}u'(r)$ is nonincreasing in $[0,R']$ and nondecreasing in $[R',R]$. Since $u'(0) = u'(R) = 0$, we obtain that $u' \le 0$ in $[0,R]$. Using the fact that $u(0) = u(R) = 0$, we deduce that $u = 0$ in $[0,R]$ and we get a contradiction. Now (2.1) implies that $v' > 0$ in $(0,R]$. Let $R' \in (0,R)$ be such that $v(R') = 0$. Using the first equation in (1.1), we deduce that $r^{n-1}u'(r)$ is decreasing in $[0,R']$ and increasing in $[R',R]$. Since $u'(0) = u'(R) = 0$, we obtain $u' < 0$ in $(0,R)$. □

Lemma 2.2. *Assume that $n \ge 1$ and $p,q > 0$. Let $\alpha,\beta > 0$ be fixed. If $(u,v) \in (C^2(\mathbb{R}^n))^2$ is a radial solution of*

$$\begin{gathered} \Delta u = |v|^{q-1}v, \quad r > 0, \\ \Delta v = |u|^p, \quad r > 0, \\ u(0) = \alpha, \qquad v(0) = -\beta, \qquad u'(0) = v'(0) = 0 \end{gathered} \tag{2.2}$$

such that $uu' < 0$ on $(0,\infty)$, then $v < 0$ on $(0,\infty)$.

Proof. We have $0 < u \le \alpha$ on $[0,\infty)$. Therefore

$$r^{n-1}v'(r) = \int_0^r s^{n-1}u(s)^p ds > 0 \quad \text{for } r > 0. \tag{2.3}$$

Assume that the conclusion of the lemma is false. Then (2.3) implies that there exist $a,b > 0$ such that

$$v(r) \ge a \quad \text{for } r \ge b. \tag{2.4}$$

We deduce that

$$\left(r^{n-1}u'(r)\right)' \ge a^q r^{n-1} \quad \text{for } r \ge b, \tag{2.5}$$

hence

$$r^{n-1}u'(r) \ge a^q \frac{r^n - b^n}{n} + b^{n-1}u'(b) \quad \text{for } r \ge b, \tag{2.6}$$

which implies that $u'(r) > 0$ for r large and we reach a contradiction. □

Now we give a lemma which is needed in the proof of Theorem 1.3.

Lemma 2.3. *Assume that $n \ge 1$ and $p,q > 0$. Let $\alpha,\beta > 0$ be fixed. Assume that for some $a > 0$, $(u,v) \in (C^2(\overline{B}_a))^2$ is a radial solution of*

$$\begin{gathered} \Delta u = |v|^{q-1}v \quad \textit{in } [0,a], \\ \Delta v = |u|^p \quad \textit{in } [0,a], \\ u(0) = \alpha, \qquad v(0) = -\beta, \qquad u'(0) = v'(0) = 0 \end{gathered} \tag{2.7}$$

such that $uu' < 0$ *on* $(0,a)$. *Then*

$$|v(r)| \le d \max\left(\beta, \alpha^{(p+1)/(q+1)}\right), \quad 0 \le r \le a, \tag{2.8}$$

where

$$d = \left(1 + \frac{q+1}{p+1}\right)^{1/(q+1)}. \tag{2.9}$$

Proof. We have $0 < u \le \alpha$ on $[0,a)$. As in Lemma 2.2 we deduce that $v' > 0$ on $(0,a]$. We have

$$\int_0^r (v'\Delta u + u'\Delta v)ds = \int_0^r \left(|v|^{q-1}vv' + u^p u'\right)ds \tag{2.10}$$

for $r \in [0,a]$. Since

$$\begin{aligned}\int_0^r (v'\Delta u + u'\Delta v)ds &= \int_0^r (u'v')'ds + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}ds \\ &= u'(r)v'(r) + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}ds,\end{aligned} \tag{2.11}$$

$$\int_0^r \left(|v|^{q-1}vv' + u^p u'\right)ds = \frac{|v(r)|^{q+1}}{q+1} + \frac{u(r)^{p+1}}{p+1} - \frac{\beta^{q+1}}{q+1} - \frac{\alpha^{p+1}}{p+1}, \tag{2.12}$$

we obtain

$$\frac{|v(r)|^{q+1}}{q+1} + \frac{u(r)^{p+1}}{p+1} = \frac{\beta^{q+1}}{q+1} + \frac{\alpha^{p+1}}{p+1} + u'(r)v'(r) + 2(n-1)\int_0^r \frac{u'(s)v'(s)}{s}ds \tag{2.13}$$

for $r \in [0,a]$, which implies that

$$|v(r)|^{q+1} \le \beta^{q+1} + \frac{q+1}{p+1}\alpha^{p+1}, \quad 0 \le r \le a, \tag{2.14}$$

and the lemma follows. □

3. Proof of Theorem 1.1

(i) Let (u,v) and (w,z) be two nontrivial radial solutions of (1.1). Let s and t be defined by

$$s = 2\frac{q+1}{pq-1}, \qquad t = 2\frac{p+1}{pq-1}. \tag{3.1}$$

For $\lambda > 0$ we set

$$\widetilde{w}(r) = \lambda^s w(\lambda r), \quad \widetilde{z}(r) = \lambda^t z(\lambda r), \quad 0 \le r \le \frac{R}{\lambda}. \tag{3.2}$$

By Lemma 2.1, $\widetilde{w} > 0$ on $[0, R/\lambda)$ and then we have

$$\begin{aligned} \Delta\widetilde{w}(r) &= |\widetilde{z}(r)|^{q-1} z(r), \quad 0 \le r \le \frac{R}{\lambda}, \\ \Delta\widetilde{z}(r) &= \widetilde{w}(r)^p, \quad 0 \le r \le \frac{R}{\lambda}, \\ \widetilde{w}\left(\frac{R}{\lambda}\right) &= \widetilde{w}'\left(\frac{R}{\lambda}\right) = 0. \end{aligned} \tag{3.3}$$

Choose λ such that $\lambda^s w(0) = u(0)$. Then we have

$$\widetilde{w}(0) = u(0). \tag{3.4}$$

We want to show that

$$\widetilde{z}(0) = v(0). \tag{3.5}$$

Suppose that $\widetilde{z}(0) < v(0)$. If there exists $a \in (0, \min(R, R/\lambda)]$ such that $\widetilde{z} - v < 0$ on $[0, a)$ and $(\widetilde{z} - v)(a) = 0$, then $\Delta(\widetilde{w} - u) < 0$ on $[0, a)$. Equation (3.4) and the maximum principle imply that $\widetilde{w} - u < 0$ on $(0, a]$. Therefore $\Delta(\widetilde{z} - v) < 0$ on $(0, a]$ and the maximum principle implies that $\widetilde{z} - v > (\widetilde{z} - v)(a) = 0$ on $[0, a)$, a contradiction. Thus $\widetilde{z} - v < 0$ on $[0, \min(R, R/\lambda)]$. Then, as before, we show that $\widetilde{w} - u < 0$ on $(0, \min(R, R/\lambda)]$. Since

$$(\widetilde{w} - u)\left(\min\left(R, \frac{R}{\lambda}\right)\right) = \begin{cases} -u\left(\frac{R}{\lambda}\right) & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ \widetilde{w}(R) & \text{if } \lambda < 1, \end{cases} \tag{3.6}$$

we deduce that $\lambda > 1$ with the help of Lemma 2.1. Now using the fact that $r^{n-1}(\widetilde{w} - u)'(r)$ is decreasing in $[0, R/\lambda]$, we get $(\widetilde{w} - u)'(R/\lambda) < 0$. Since $(\widetilde{w} - u)'(R/\lambda) = -u'(R/\lambda) > 0$ by Lemma 2.1, we again obtain a contradiction. The case $\widetilde{z}(0) > v(0)$ can be handled in the same way. Thus (3.5) is proved.

Now we define the functions U, W, F, and G_n by

$$\begin{aligned} U(r) &= \big(u(r), v(r)\big), \quad 0 \le r \le R, \\ W(r) &= \big(\widetilde{w}(r), \widetilde{z}(r)\big), \quad 0 \le r \le \frac{R}{\lambda}, \\ F(x, y) &= \big(|y|^{q-1} y, x^p\big), \quad x \ge 0,\ y \in \mathbb{R}, \end{aligned} \tag{3.7}$$

$$G_n(r, s) = \begin{cases} r - s & \text{if } n = 1, \\ s \ln\left(\frac{r}{s}\right) & \text{if } n = 2, \\ \frac{s}{n-2}\left(1 - \left(\frac{s}{r}\right)^{n-2}\right) & \text{if } n \ge 3 \end{cases} \tag{3.8}$$

for $0 \le s \le r$. Using (3.4), (3.5), and the fact that $u'(0) = \widetilde{w}'(0) = v'(0) = \widetilde{z}'(0) = 0$, we easily obtain

$$U(r) - W(r) = \int_0^r G_n(r,s)\big(F(U(s)) - F(W(s))\big)ds \tag{3.9}$$

for $r \in [0, \min(R, R/\lambda)]$. When $p \ge 1$, F is locally Lipschitz continuous, and using Gronwall's lemma we obtain $U = W$ on $[0, \min(R, R/\lambda)]$. When $p \in (0,1)$, let $a \in (0, \min(R, R/\lambda))$ be fixed. Then $u(0) \ge u(r) \ge u(a) > 0$, $\widetilde{w}(0) = u(0) \ge \widetilde{w}(r) \ge \widetilde{w}(a) > 0$ for $r \in [0,a]$. Since F is locally Lipschitz continuous on $(0,+\infty) \times \mathbb{R}$, as before we obtain $U = W$ on $[0,a]$. By continuity we get $U = W$ on $[0, \min(R, R/\lambda)]$. Now we deduce that $\lambda = 1$ and thus $(u,v) = (w,z)$ on $[0,R]$.

(ii) Let (u,v) be a nontrivial radial solution of problem (1.1). Then, for any $\theta > 0$, $(w,z) = (\theta^q u, \theta v)$ is a nontrivial radial solution of problem (1.1). Now let (w,z) be a nontrivial radial solution of (1.1). Choose $\theta > 0$ such that $\theta^q u(0) = w(0)$ and define $\widetilde{w} = \theta^q u$, $\widetilde{z} = \theta v$. Then $(\widetilde{w}, \widetilde{z})$ is a nontrivial radial solution of (1.1) such that $\widetilde{w}(0) = w(0)$. Arguing as in part (i), we show that $\widetilde{z}(0) = z(0)$ and that $(\widetilde{w}, \widetilde{z}) = (w,z)$.

Remark 3.1. Our technique also applies when there is a homogeneous dependence on the radius $|x|$. More precisely, for $p > 0$, $q \ge 1$, and $pq \ne 1$, the following system

$$\begin{aligned} \Delta u &= |x|^\mu |v|^{q-1} v \quad \text{in } B_R, \\ \Delta v &= |x|^\nu |u|^p \quad \text{in } B_R, \\ u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B_R, \end{aligned} \tag{3.10}$$

where $\mu, \nu \ge 0$, has at most one nontrivial radial solution (u,v). Indeed, the arguments are the same with s and t in (2.1) replaced by

$$s = \frac{2(q+1) + \nu + q\mu}{pq-1}, \qquad t = \frac{2(p+1) + \mu + p\nu}{pq-1}. \tag{3.11}$$

Now let $p > 0$, $q \ge 1$ with $pq = 1$. Assume that problem (3.10) has a nontrivial radial solution (u,v). Then all nontrivial radial solutions are given by $(\theta^q u, \theta v)$, where $\theta > 0$ is an arbitrary constant.

4. Proof of Theorem 1.2

(i) Let $(u,v) \in (C^2(\overline{B}_R))^2$ be a solution of problem (1.1) such that $u \ge 0$ in B_R. We have $x \cdot \nu(x) = R$ for all $x \in \partial B_R$. Multiplying the first equation in (1.1) by $x \cdot \nabla v$ and integrating over B_R, we get

$$\int_{B_R} (x \cdot \nabla v) \Delta u \, dx = \int_{B_R} (x \cdot \nabla v) |v|^{q-1} v \, dx. \tag{4.1}$$

Integrating by parts, we obtain

$$\int_{B_R} (x \cdot \nabla v)|v|^{q-1} v \, dx = -\frac{n}{q+1} \int_{B_R} |v|^{q+1} dx + \frac{R}{q+1} \int_{\partial B_R} |v|^{q+1} d\sigma. \tag{4.2}$$

Similarly we get

$$\int_{B_R} (x \cdot \nabla u)\Delta v\, dx = \int_{B_R} (x \cdot \nabla u)u^p\, dx = -\frac{n}{p+1}\int_{B_R} u^{p+1} dx. \tag{4.3}$$

Now we have

$$\int_{B_R} \big((x \cdot \nabla v)\Delta u + (x \cdot \nabla u)\Delta v\big)\, dx = (n-2)\int_{B_R} \nabla u \cdot \nabla v\, dx. \tag{4.4}$$

Then we deduce that

$$\frac{R}{q+1}\int_{\partial B_R} |v|^{q+1} d\sigma = \frac{n}{q+1}\int_{B_R} |v|^{q+1} dx + \frac{n}{p+1}\int_{B_R} u^{p+1} dx + (n-2)\int_{B_R} \nabla u \cdot \nabla v\, dx. \tag{4.5}$$

Since

$$\begin{aligned} \int_{B_R} \nabla u \cdot \nabla v\, dx &= -\int_{B_R} v\Delta u\, dx = -\int_{B_R} |v|^{q+1} dx, \\ \int_{B_R} \nabla u \cdot \nabla v\, dx &= -\int_{B_R} u\Delta v\, dx = -\int_{B_R} u^{p+1} dx, \end{aligned} \tag{4.6}$$

we can write

$$\frac{R}{q+1}\int_{\partial B_R} |v|^{q+1} d\sigma = n\left(\frac{1}{p+1} + \frac{1}{q+1} - \frac{n-2}{n}\right)\int_{B_R} |v|^{q+1} dx. \tag{4.7}$$

Using (1.4) we deduce that $v = 0$ on ∂B_R. The maximum principle implies that $v \le 0$ in B_R. Therefore $\Delta u \le 0$ in B_R. The Hopf boundary point lemma implies that $u = 0$ in B_R and (i) is proved.

(ii) follows from (i) and Lemma 2.1.

Remark 4.1. Clearly Theorem 1.2(i) can be extended to more general domains and more general nonlinearities as in [2, 11, 12] and Theorem 1.2(ii) can be extended to more general nonlinearities.

5. Proof of Theorem 1.3

We will use a two-dimensional shooting argument for the ordinary differential equations associated to radial solutions of (1.1) [3, 5, 7, 15, 16]. We consider the one-dimensional (singular if $n \ge 2$) initial value problem (2.2) where $\alpha > 0$, $\beta > 0$.

We will need a series of lemmas. We begin with a standard local existence and uniqueness result.

Lemma 5.1. *For any $\alpha > 0$, $\beta > 0$ there exists $T = T(\alpha,\beta) > 0$ such that problem (2.2) on $[0,T]$ has a unique solution $(u,v) \in (C^2[0,T])^2$.*

Proof. Let $\alpha,\beta > 0$ be given. Choose $T = T(\alpha,\beta) > 0$ such that

$$T = \min\left(\left(\frac{n\alpha}{\beta^q}\right)^{1/2}, \left(\frac{n\beta}{\alpha^p}\right)^{1/2}\right), \tag{5.1}$$

and consider the set of functions

$$Z = \left\{(u,v) \in \left(C[0,T]\right)^2;\ \frac{\alpha}{2} \le u(r) \le \alpha,\ -\beta \le v(r) \le -\frac{\beta}{2} \text{ for } 0 \le r \le T\right\}. \tag{5.2}$$

Clearly Z is a bounded closed convex subset of the Banach space $(C[0,T])^2$ endowed with the norm $\|(u,v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Define

$$L(u,v)(r) = \left(\alpha + \int_0^r G_n(r,s)\,|\,v(s)\,|^{q-1}v(s)ds, -\beta + \int_0^r G_n(r,s)\,|\,u(s)\,|^p ds\right) \tag{5.3}$$

for $r \in [0,T]$ and $(u,v) \in (C[0,T])^2$, where G_n is defined in (3.8). It is easily verified that L is a compact operator mapping Z into itself, and so there exists $(u,v) \in Z$ such that $(u,v) = L(u,v)$ by the Schauder fixed point theorem. Clearly $(u,v) \in (C^2[0,T])^2$ and (u,v) is a solution of (2.2) on $[0,T]$. Since the right-hand side in (2.2) is Lipschitz continuous in $(u,v) \in [\alpha/2,\alpha] \times [-\beta,-\beta/2]$, the uniqueness follows. □

Remark 5.2. Notice that $u(r) > 0$ and $v(r) < 0$ for $r \in [0,T]$. Then direct integration of the system (2.2) implies that $u' < 0$ and $v' > 0$ in $(0,T]$.

In view of Lemma 5.1, for any $\alpha,\beta > 0$ problem (2.2) has a unique local solution: let $[0,R_{\alpha,\beta})$ denote the maximum interval of existence of that solution ($R_{\alpha,\beta} = +\infty$ possibly). If $0 < p < 1$, the uniqueness of the solution could fail at any point r where $u(r) = 0$. In this case, $R_{\alpha,\beta}$ could also depend on the particular solution itself. Define

$$P_{\alpha,\beta} = \{s \in (0,R_{\alpha,\beta});\ u(\alpha,\beta,r)u'(\alpha,\beta,r) < 0\ \forall r \in (0,s]\}, \tag{5.4}$$

where $(u(\alpha,\beta,\cdot), v(\alpha,\beta,\cdot))$ is a solution of (2.2) in $[0,R_{\alpha,\beta})$. $P_{\alpha,\beta} \neq \emptyset$ by Remark 5.2. Set

$$r_{\alpha,\beta} = \sup P_{\alpha,\beta}. \tag{5.5}$$

Notice that the solution is unique on $[0,r_{\alpha,\beta}]$, so $r_{\alpha,\beta}$ depends only on α, β.

Lemma 5.3. *$u'(\alpha,\beta,r) < 0$ for $r \in (0,r_{\alpha,\beta})$ and $v'(\alpha,\beta,r) > 0$ for $r \in (0,R_{\alpha,\beta})$.*

Proof. The first assertion follows from the definition of $r_{\alpha,\beta}$. Since $u(\alpha,\beta,r) > 0$ for $r \in [0,r_{\alpha,\beta})$, integrating the second equation in (2.2) from 0 to $r \in (0,R_{\alpha,\beta})$ we obtain $v'(\alpha,\beta, r) > 0$ for $r \in (0,R_{\alpha,\beta})$. □

Lemma 5.4. *For any $\alpha,\beta > 0$, $r_{\alpha,\beta} < \infty$.*

Proof. Assume that $r_{\alpha,\beta} = \infty$. We easily get a contradiction when $n = 1$ or 2. Now if $n \ge 3$, we set $z = -v$. By Lemma 2.2, $z > 0$ on $[0,\infty)$ and we have

$$\begin{aligned} -\Delta u &= z^q, \quad r > 0,\\ -\Delta z &= u^p, \quad r > 0. \end{aligned} \tag{5.6}$$

Since p, q satisfy (1.5), we obtain a contradiction with the help of the nonexistence results established in [9, 10, 13, 14]. □

LEMMA 5.5. *For any* $a \in [T(\alpha,\beta), r_{\alpha,\beta})$, *there exists* $b = b(\alpha,\beta,a) > 0$ *such that the maximal extension of* (u,v) *includes the interval* $[0,a+b]$. *Moreover,*

$$b(\alpha,\beta,a) = \frac{m(\alpha,\beta)}{a+\sqrt{a^2+m(\alpha,\beta)}}, \tag{5.7}$$

where

$$m(\alpha,\beta) = \min\left(\frac{n\beta}{2^{p-1}\alpha^p}, \frac{n\alpha}{2^{q-1}d^q\left(\max\left(\beta,\alpha^{(p+1)/(q+1)}\right)\right)^q}\right), \tag{5.8}$$

with d given in Lemma 2.3.

Proof. Lemma 5.5 is essentially a local existence result, with initial data $u(a)$, $v(a)$, $u'(a)$, $v'(a)$ at $r = a$. Let

$$W = \left\{(u,v) \in \left(C[a,a+b]\right)^2;\ |u(r)-u(a)| \le \alpha,\ 0 \le v(r)-v(a) \le \beta \text{ for } a \le r \le a+b\right\}, \tag{5.9}$$

where $b = b(\alpha,\beta,a)$ is given in the lemma. W is a bounded closed convex subset of the Banach space $(C[a,a+b])^2$ equipped with the norm $\|(u,v)\| = \max(\|u\|_\infty, \|v\|_\infty)$. Consider the mapping $S(u,v) = (S_1(u,v), S_2(u,v))$ on $(C[a,a+b])^2$ given by

$$\begin{aligned} S_1(u,v)(r) &= u(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1}|v(s)|^{q-1}v(s)ds, \\ S_2(u,v)(r) &= v(a) + \int_a^r \frac{dt}{t^{n-1}} \int_0^t s^{n-1}|u(s)|^p ds \end{aligned} \tag{5.10}$$

for $a \le r \le a+b$, where we also denote by u, v the unique solution of (2.2) on $[0,a]$. Let $(u,v) \in W$. Using Lemma 5.3, we have

$$|u(s)| \le u(a) + \alpha \le 2\alpha, \quad s \in [a,a+b]. \tag{5.11}$$

Therefore we get

$$0 \le S_2(u,v)(r) - v(a) \le 2^{p-1}\alpha^p \frac{r^2-a^2}{n} \le \beta, \quad r \in [a,a+b]. \tag{5.12}$$

By Lemma 2.3 we have

$$|v(s)| \le |v(a)| + \beta \le 2d\max\left(\beta,\alpha^{(p+1)/(q+1)}\right), \quad s \in [a,a+b]. \tag{5.13}$$

Therefore for $a \le r \le a+b$, we obtain

$$|S_1(u,v)(r) - u(a)| \le 2^{q-1}d^q\left(\max\left(\beta,\alpha^{(p+1)/(q+1)}\right)\right)^q \frac{r^2-a^2}{n} \le \alpha. \tag{5.14}$$

We have thus proved that $S(W) \subset W$. Since S is a compact operator, there exists $(u,v) \in W$ such that $(u,v) = S(u,v)$ by the Schauder fixed point theorem. Clearly $(u,v) \in (C^2[a,a+b])^2$ and (u,v) is a solution of (2.2) on $[a,a+b]$ which extends the solution (u,v) on $[0,a]$. □

Lemma 5.6. *For any $\alpha,\beta > 0$,*

$$R_{\alpha,\beta} \geq r_{\alpha,\beta} + \frac{m(\alpha,\beta)}{r_{\alpha,\beta} + \sqrt{r_{\alpha,\beta}^2 + m(\alpha,\beta)}}. \tag{5.15}$$

Proof. By Lemma 5.5, for any $a \in (T(\alpha,\beta), r_{\alpha,\beta})$ we have

$$R_{\alpha,\beta} > a + \frac{m(\alpha,\beta)}{a + \sqrt{a^2 + m(\alpha,\beta)}}. \tag{5.16}$$

The lemma follows by letting $a \to r_{\alpha,\beta}$. □

Proposition 5.7. *For any $\alpha > 0$, there exists a unique $\beta > 0$ such that $u(\alpha,\beta,r_{\alpha,\beta}) = u'(\alpha,\beta, r_{\alpha,\beta}) = 0$.*

Proof. We first prove the uniqueness. Let $\alpha > 0$ be fixed. Suppose that there exist $\beta > \gamma > 0$ such that $u(\alpha,\beta,r_{\alpha,\beta}) = u'(\alpha,\beta,r_{\alpha,\beta}) = u(\alpha,\gamma,r_{\alpha,\gamma}) = u'(\alpha,\gamma,r_{\alpha,\gamma}) = 0$. Using the same arguments as in the proof of (3.5) we obtain a contradiction.

Now we prove the existence. Suppose that there exists $\alpha > 0$ such that for any $\beta > 0$ $u(\alpha,\beta,r_{\alpha,\beta}) > 0$ or $u'(\alpha,\beta,r_{\alpha,\beta}) < 0$. Define the sets

$$\begin{aligned} B &= \{\beta > 0;\ u(\alpha,\beta,r_{\alpha,\beta}) = 0,\ u'(\alpha,\beta,r_{\alpha,\beta}) < 0\}, \\ C &= \{\beta > 0;\ u(\alpha,\beta,r_{\alpha,\beta}) > 0,\ u'(\alpha,\beta,r_{\alpha,\beta}) = 0\}. \end{aligned} \tag{5.17}$$

□

The proof of the proposition is completed by using the next two lemmas which contradict the fact that

$$(0,+\infty) = B \cup C. \tag{5.18}$$

Lemma 5.8. (i) *Suppose $B \neq \emptyset$. Then there exists $m > 0$ such that $m \leq \inf B$.*

(ii) *Suppose $C \neq \emptyset$. Then there exists $M > 0$ such that $M \geq \sup C$.*

Lemma 5.9. *B and C are open.*

Proof of Lemma 5.8. We have

$$u(\alpha,\beta,r) = \alpha + \int_0^r G_n(r,s)\,|v(\alpha,\beta,s)|^{q-1} v(\alpha,\beta,s)\,ds, \quad 0 \leq r < R_{\alpha,\beta}, \tag{5.19}$$

$$v(\alpha,\beta,r) = -\beta + \int_0^r G_n(r,s)\,|u(\alpha,\beta,s)|^{p}\,ds, \quad 0 \leq r < R_{\alpha,\beta}. \tag{5.20}$$

(i) Let $\beta \in B$. Assume first that $v(\alpha,\beta,\cdot) < 0$ on $[0, r_{\alpha,\beta})$. Then Lemma 5.3 and (5.19) imply

$$r_{\alpha,\beta} \geq \left(\frac{2n\alpha}{\beta^q}\right)^{1/2}. \tag{5.21}$$

Now, if there exists $s_{\alpha,\beta} \in [0, r_{\alpha,\beta})$ such that $v(\alpha,\beta,s_{\alpha,\beta}) = 0$, Lemma 5.3 implies that $-\beta \leq v(\alpha,\beta,\cdot) < 0$ in $[0, s_{\alpha,\beta})$ and $v(\alpha,\beta,\cdot) > 0$ in $(s_{\alpha,\beta}, r_{\alpha,\beta}]$. Then from (5.19) we get

$$\begin{aligned}\alpha &= -\int_0^{r_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)\,|\,v(\alpha,\beta,s)\,|^{q-1}v(\alpha,\beta,s)ds \\ &\leq \int_0^{s_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)\,|\,v(\alpha,\beta,s)\,|^{q}ds \leq \beta^q\int_0^{s_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)ds \leq \beta^q\frac{r_{\alpha,\beta}^2}{2n},\end{aligned} \tag{5.22}$$

and (5.21) still holds.

Suppose that $\inf B = 0$ and let (β_j) be a sequence in B decreasing to zero. Then $r_{\alpha,\beta_j} \to +\infty$ by (5.21). Let $r > 0$ be fixed. We can assume that $r_{\alpha,\beta_j} > r$ for all j. If $v(\alpha,\beta_j,s) < 0$ for $s \in [0,r]$, we have

$$u(\alpha,\beta_j,r) = \alpha - \int_0^r G_n(r,s)\,|\,v(\alpha,\beta_j,s)\,|^{q}ds \geq \alpha - \frac{r^2\beta_j^q}{2n}. \tag{5.23}$$

If $s_{\alpha,\beta_j} < r$, we have

$$\begin{aligned}u(\alpha,\beta_j,r) &= \alpha - \int_0^{s_{\alpha,\beta_j}} G_n(r,s)\,|\,v(\alpha,\beta_j,s)\,|^{q}ds + \int_{s_{\alpha,\beta_j}}^r G_n(r,s)v(\alpha,\beta_j,s)^q ds \\ &\geq \alpha - \int_0^{s_{\alpha,\beta_j}} G_n(r,s)\,|\,v(\alpha,\beta_j,s)\,|^{q}ds \\ &\geq \alpha - \beta_j^q\int_0^{s_{\alpha,\beta_j}} G_n(r,s)ds \geq \alpha - \frac{r^2\beta_j^q}{2n}.\end{aligned} \tag{5.24}$$

Therefore using Lemma 5.3 we obtain

$$u(\alpha,\beta_j,s) \geq \alpha - \frac{r^2\beta_j^q}{2n} \quad \text{for } s \in [0,r], \tag{5.25}$$

from which we deduce that

$$u(\alpha,\beta_j,s) \geq \frac{\alpha}{2} \tag{5.26}$$

for $s \in [0,r]$ and j large. From (5.20) we get

$$v(\alpha,\beta_j,r) \geq -\beta_j + \frac{r^2\alpha^p}{2^{p+1}n} \tag{5.27}$$

for j large. Thus if we choose r such that

$$-\beta_j + \frac{r^2\alpha^p}{2^{p+1}n} \geq 1, \tag{5.28}$$

using Lemma 5.3 we get

$$v(\alpha,\beta_j,s) \geq 1 \tag{5.29}$$

for $r \leq s \leq r_{\alpha,\beta_j}$ and j large. We also have

$$-\beta_j \leq v(\alpha,\beta_j,s) \leq -\beta_j + \frac{r^2\alpha^p}{2n} \tag{5.30}$$

for $s \in [0,r]$. Therefore there exists $c > 0$ such that

$$|v(\alpha,\beta_j,s)| \leq c \tag{5.31}$$

for $s \in [0,r]$ and all j. There exists $k > 0$ such that

$$\int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,ds \geq k r_{\alpha,\beta_j}^2 \tag{5.32}$$

for j large. Now we write

$$\begin{aligned}
\alpha &= -\int_0^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,|v(\alpha,\beta_j,s)|^{q-1}v(\alpha,\beta_j,s)\,ds \\
&= -\int_0^{r} G_n(r_{\alpha,\beta_j},s)\,|v(\alpha,\beta_j,s)|^{q-1}v(\alpha,\beta_j,s)\,ds \\
&\quad -\int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,v(\alpha,\beta_j,s)^q\,ds \\
&\leq c^q\int_0^{r} G_n(r_{\alpha,\beta_j},s)\,ds - \int_r^{r_{\alpha,\beta_j}} G_n(r_{\alpha,\beta_j},s)\,ds \\
&\leq c^q r r_{\alpha,\beta_j} - k r_{\alpha,\beta_j}^2
\end{aligned} \tag{5.33}$$

for j large, where we have used the fact that $G_n(r_{\alpha,\beta_j},s) \leq r_{\alpha,\beta_j} - s$ for $0 \leq s \leq r_{\alpha,\beta_j}$. Since the last term above tends to $-\infty$, we get a contradiction.

(ii) Let $\beta \in C$. We claim that $v(\alpha,\beta,r_{\alpha,\beta}) > 0$. If not, by Lemma 5.3 we have $\Delta u(\alpha,\beta,\cdot) < 0$ on $[0,r_{\alpha,\beta})$ for some $\beta \in C$. Since $u'(\alpha,\beta,0) = 0$, we obtain $u'(\alpha,\beta,r_{\alpha,\beta}) < 0$, a contradiction. Therefore (5.20) implies

$$\beta < \int_0^{r_{\alpha,\beta}} G_n(r_{\alpha,\beta},s)\,u(\alpha,\beta,s)^p\,ds \tag{5.34}$$

for $\beta \in C$. Suppose that $\sup C = +\infty$ and let (β_j) be a sequence in C increasing to $+\infty$. Since $0 < u(\alpha,\beta_j,r) \leq \alpha$ for $r \in [0,r_{\alpha,\beta_j}]$, (5.34) implies that $r_{\alpha,\beta_j} \to +\infty$ as $j \to +\infty$. Then we can assume that $r_{\alpha,\beta_j} \geq 1$ and that $\alpha^p \leq \beta_j$ for all j. From (5.20) we get

$$-\beta_j \leq v(\alpha,\beta_j,r) \leq -\frac{2n-1}{2n}\beta_j \leq -\frac{\beta_j}{2} \quad \text{for } r \in [0,1], \tag{5.35}$$

and using (5.19) we deduce that $u(\alpha,\beta_j,1) \leq \alpha - \beta_j^q/n2^{q+1}$. But then $u(\alpha,\beta_j,1) < 0$ for j large and we reach a contradiction. □

Remark 5.10. The proof above shows that, when $\beta \in C$, there exists $s_{\alpha,\beta} \in (0, r_{\alpha,\beta})$ such that $v(\alpha,\beta,\cdot) < 0$ on $[0, s_{\alpha,\beta})$ and $v(\alpha,\beta,\cdot) > 0$ on $(s_{\alpha,\beta}, r_{\alpha,\beta}]$. When $\beta \in B$, $s_{\alpha,\beta}$ may not exist.

Proof of Lemma 5.9

Case 1 ($p \geq 1$). Then the right-hand side of (2.2) is Lipschitz continuous. Let $\beta \in B$. We have $u(\alpha,\beta,r_{\alpha,\beta}) = 0$ and $u'(\alpha,\beta,r_{\alpha,\beta}) < 0$. Therefore we can find $\varepsilon > 0$ such that

$$u(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) < 0, \qquad u'(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) < 0. \tag{5.36}$$

But then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) < 0, \qquad u'(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) < 0 \tag{5.37}$$

for $|\gamma - \beta| < \eta$. The first inequality in (5.37) implies that there exists $x \in (0, r_{\alpha,\beta}+\varepsilon)$ such that $u(\alpha,\gamma,x) = 0$ and $u(\alpha,\gamma,r) > 0$ for $r \in [0,x)$. $\Delta v(\alpha,\gamma,r) > 0$ for $r \in [0,x)$ and $\Delta v(\alpha,\gamma,r) \geq 0$ for $r \in [x, r_{\alpha,\beta}+\varepsilon]$. Then $v'(\alpha,\gamma,r) > 0$ for $r \in (0, r_{\alpha,\beta}+\varepsilon]$ and $v(\alpha,\gamma,\cdot)$ is increasing on $[0, r_{\alpha,\beta}+\varepsilon]$. We deduce that $\Delta u(\alpha,\gamma,\cdot)$ is increasing on $[0, r_{\alpha,\beta}+\varepsilon]$. If $\Delta u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) \leq 0$, then $u'(\alpha,\gamma,r) < 0$ for $r \in (0, r_{\alpha,\beta}+\varepsilon]$. If $\Delta u(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) > 0$, then there exists $s_{\alpha,\gamma} \in (0, r_{\alpha,\beta}+\varepsilon)$ such that $\Delta u(\alpha,\gamma,\cdot) < 0$ in $[0, s_{\alpha,\gamma})$ and $\Delta u(\alpha,\gamma,\cdot) > 0$ in $(s_{\alpha,\gamma}, r_{\alpha,\beta}+\varepsilon]$. We deduce that $u'(\alpha,\gamma,\cdot)$ is decreasing (resp., increasing) in $[0, s_{\alpha,\gamma}]$ (resp., $[s_{\alpha,\gamma}, r_{\alpha,\beta}+\varepsilon]$). Since $u'(\alpha,\gamma,0) = 0$, the second inequality in (5.37) implies that $u'(\alpha,\gamma,r) < 0$ for $r \in (0, r_{\alpha,\beta}+\varepsilon]$. Therefore $x = r_{\alpha,\gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset B$. Thus B is open. Now let $\beta \in C$. We have $u(\alpha,\beta,r_{\alpha,\beta}) > 0$ and $u'(\alpha,\beta,r_{\alpha,\beta}) = 0$. By Remark 5.10, we have $v(\alpha,\beta,r_{\alpha,\beta}) > 0$, hence $\Delta u(\alpha,\beta,r_{\alpha,\beta}) = u''(\alpha,\beta,r_{\alpha,\beta}) > 0$. Therefore we can find $\varepsilon > 0$ such that

$$u(\alpha,\beta,r) > 0, \quad r \in [0, r_{\alpha,\beta}+\varepsilon], \qquad u'(\alpha,\beta,r_{\alpha,\beta}+\varepsilon) > 0. \tag{5.38}$$

Then by continuous dependence on initial data, there exists $\eta > 0$ such that

$$u(\alpha,\gamma,r) > 0, \quad r \in [0, r_{\alpha,\beta}+\varepsilon], \qquad u'(\alpha,\gamma,r_{\alpha,\beta}+\varepsilon) > 0 \tag{5.39}$$

for $|\gamma - \beta| < \eta$. The second inequality in (5.39) implies that there exists $x \in (0, r_{\alpha,\beta}+\varepsilon)$ such that $u'(\alpha,\gamma,x) = 0$ and $u'(\alpha,\gamma,r) < 0$ for $r \in (0,x)$. Therefore $x = r_{\alpha,\gamma}$ for $|\gamma - \beta| < \eta$ and $(\beta - \eta, \beta + \eta) \subset C$. Thus C is open.

Case 2 ($0 < p < 1$). We first show that C is open. Indeed let $\beta \in C$. Since $u(\alpha,\beta,r) > 0$ for $r \in [0, r_{\alpha,\beta}]$, the system (2.2) is Lipschitz continuous in u and v when u is in a neighborhood of the interval $[u(\alpha,\beta,r_{\alpha,\beta}), \alpha]$ in $(0,\infty)$, and the solution $u(\alpha,\beta,\cdot)$, $v(\alpha,\beta,\cdot)$ can be uniquely extended to $[0, r_{\alpha,\beta}+t]$ for some $t > 0$, with $u(\alpha,\beta,r) > 0$ for $r \in [0, r_{\alpha,\beta}+t]$. Then we can argue as in Case 1. Now we show that B is open. As in [15], this case is much more difficult. We begin with the following two steps. Let $\beta \in B$.

Step 1. There exists $c > 0$ and $\eta > 0$ such that when $|\beta - \gamma| < \eta$, the solutions $u(\alpha,\gamma,\cdot)$, $v(\alpha,\gamma,\cdot)$, and $u(\alpha,\beta,\cdot)$, $v(\alpha,\beta,\cdot)$ are defined on $[0, r_{\alpha,\beta}+c]$.

By Lemma 5.6, $u(\alpha,\beta,\cdot)$, $v(\alpha,\beta,\cdot)$ can be extended to the interval $[0,r_{\alpha,\beta}+b(\alpha,\beta,r_{\alpha,\beta}))$ where

$$b(\alpha,\beta,r_{\alpha,\beta}) = \frac{m(\alpha,\beta)}{r_{\alpha,\beta}+\sqrt{r_{\alpha,\beta}^2+m(\alpha,\beta)}}. \tag{5.40}$$

Fix $\omega \in (0,r_{\alpha,\beta}-T(\alpha,\beta))$ and $\mu = r_{\alpha,\beta}-\omega$. Then $T(\alpha,\beta) < \mu < r_{\alpha,\beta}$ and by Lemma 5.3

$$0 < u(\alpha,\beta,\mu) \le u(\alpha,\beta,r) \le \alpha, \quad 0 \le r \le \mu. \tag{5.41}$$

Since the system (2.2) is Lipschitz continuous in u and v when u is in a neighborhood of the interval $[u(\alpha,\beta,\mu),\alpha]$ in $(0,\infty)$, the continuous dependence on initial data implies that there exists $\eta > 0$ such that when $|\gamma-\beta| < \eta$ the solution $u(\alpha,\gamma,\cdot)$, $v(\alpha,\gamma,\cdot)$ is defined on $[0,\mu]$ and $u(\alpha,\gamma,r) > 0$ for $r \in [0,\mu]$, $u'(\alpha,\gamma,r) < 0$ for $r \in (0,\mu]$, hence $r_{\alpha,\gamma} > \mu$. By taking η smaller if necessary, we can assume that $T(\alpha,\gamma) < \mu$, hence $T(\alpha,\gamma) < \mu < r_{\alpha,\gamma}$. By Lemma 5.5 we can extend $u(\alpha,\gamma,\cdot)$, $v(\alpha,\gamma,\cdot)$ to $[0,\mu+b(\alpha,\gamma,\mu)]$. By taking η smaller if necessary, we can assume that

$$b(\alpha,\gamma,\mu) > \frac{b(\alpha,\beta,\mu)}{2} > \frac{b(\alpha,\beta,r_{\alpha,\beta})}{2} = 2c. \tag{5.42}$$

Thus if we choose ω to satisfy also $\omega \le c$, we get

$$\mu+b(\alpha,\gamma,\mu) = r_{\alpha,\beta}-\omega+b(\alpha,\gamma,\mu) \ge r_{\alpha,\beta}+c. \tag{5.43}$$

Thus $u(\alpha,\gamma,\cdot)$, $v(\alpha,\gamma,\cdot)$ extend to the interval $[0,r_{\alpha,\beta}+c]$ and $c < b(\alpha,\beta,r_{\alpha,\beta})$ so that $u(\alpha,\beta,\cdot)$, $v(\alpha,\beta,\cdot)$ also exist on $[0,r_{\alpha,\beta}+c]$.

Step 2. We claim that there exist $\varepsilon \in (0,c)$ and $\delta \in (0,\eta)$ such that

$$\left| u'(\alpha,\gamma,r)-u'(\alpha,\beta,r_{\alpha,\beta}) \right| \le \frac{1}{2}\left| u'(\alpha,\beta,r_{\alpha,\beta}) \right| \tag{5.44}$$

(recall that $u'(\alpha,\beta,r_{\alpha,\beta}) < 0$) when $|\gamma-\beta| < \delta$ and $|r-r_{\alpha,\beta}| \le \varepsilon$. Let $\varepsilon \in (0,c)$, $|\gamma-\beta| < \eta$, and $r \in [r_{\alpha,\beta}-\varepsilon,r_{\alpha,\beta}+\varepsilon]$. By Step 1 and integration of (2.2) we have

$$\begin{aligned} u'&(\alpha,\gamma,r)-u'(\alpha,\beta,r_{\alpha,\beta}) \\ &= u'(\alpha,\gamma,r)-u'(\alpha,\beta,r)+u'(\alpha,\beta,r)-u'(\alpha,\beta,r_{\alpha,\beta}) \\ &= \left(u'(\alpha,\gamma,r_{\alpha,\beta}-\varepsilon)-u'(\alpha,\beta,r_{\alpha,\beta}-\varepsilon)\right)\frac{(r_{\alpha,\beta}-\varepsilon)^{n-1}}{r^{n-1}} \\ &\quad+\int_{r_{\alpha,\beta}-\varepsilon}^{r}\frac{s^{n-1}}{r^{n-1}}\Big(\left|v(\alpha,\gamma,s)\right|^{q-1}v(\alpha,\gamma,s)-\left|v(\alpha,\beta,s)\right|^{q-1}v(\alpha,\beta,s)\Big)ds \\ &\quad+u'(\alpha,\beta,r_{\alpha,\beta})\left(\frac{r_{\alpha,\beta}^{n-1}}{r^{n-1}}-1\right)+\int_{r_{\alpha,\beta}}^{r}\frac{s^{n-1}}{r^{n-1}}\left|v(\alpha,\beta,s)\right|^{q-1}v(\alpha,\beta,s)ds. \end{aligned} \tag{5.45}$$

We deduce that

$$
\begin{aligned}
&\left|u'(\alpha,\gamma,r)-u'(\alpha,\beta,r_{\alpha,\beta})\right| \\
&\quad\le \left|u'(\alpha,\gamma,r_{\alpha,\beta}-\varepsilon)-u'(\alpha,\beta,r_{\alpha,\beta}-\varepsilon)\right| + \left|u'(\alpha,\beta,r_{\alpha,\beta})\right| \left|\frac{r_{\alpha,\beta}^{n-1}}{r^{n-1}}-1\right| \\
&\qquad + \int_{r_{\alpha,\beta}-\varepsilon}^{r} \frac{s^{n-1}}{r^{n-1}} \left|v(\alpha,\gamma,s)\right|^{q} ds + \int_{r_{\alpha,\beta}-\varepsilon}^{r_{\alpha,\beta}} \frac{s^{n-1}}{r^{n-1}} \left|v(\alpha,\beta,s)\right|^{q} ds.
\end{aligned} \tag{5.46}
$$

The proof of Lemma 5.5 gives the following estimate for $|\gamma-\beta|<\eta$:

$$
\left|v(\alpha,\gamma,r)\right| \le 2d\max\left(\gamma,\alpha^{(p+1)/(q+1)}\right), \quad r_{\alpha,\beta}-\varepsilon \le r \le r_{\alpha,\beta}+\varepsilon. \tag{5.47}
$$

By making ε smaller if necessary we have

$$
\begin{gathered}
\int_{r_{\alpha,\beta}-\varepsilon}^{r} \frac{s^{n-1}}{r^{n-1}} \left|v(\alpha,\gamma,s)\right|^{q} ds + \int_{r_{\alpha,\beta}-\varepsilon}^{r_{\alpha,\beta}} \frac{s^{n-1}}{r^{n-1}} \left|v(\alpha,\beta,s)\right|^{q} ds \le \frac{1}{4}\left|u'(\alpha,\beta,r_{\alpha,\beta})\right|, \\
\left|\frac{r_{\alpha,\beta}^{n-1}}{r^{n-1}}-1\right| \le \frac{1}{8}
\end{gathered} \tag{5.48}
$$

for $r_{\alpha,\beta}-\varepsilon \le r \le r_{\alpha,\beta}+\varepsilon$. Then from (5.46) we obtain

$$
\left|u'(\alpha,\gamma,r)-u'(\alpha,\beta,r_{\alpha,\beta})\right| \le \left|u'(\alpha,\gamma,r_{\alpha,\beta}-\varepsilon)-u'(\alpha,\beta,r_{\alpha,\beta}-\varepsilon)\right| + \frac{3}{8}\left|u'(\alpha,\beta,r_{\alpha,\beta})\right| \tag{5.49}
$$

for $|\gamma-\beta|<\eta$ and $|r-r_{\alpha,\beta}|\le\varepsilon$. Now let ε be fixed. By continuous dependence on initial data and the fact that $u(\alpha,\beta,r)>u(\alpha,\beta,r_{\alpha,\beta}-\varepsilon)$ for $r\in[0,r_{\alpha,\beta}-\varepsilon)$, we can choose $\delta\in(0,\eta)$ such that

$$
\left|u'(\alpha,\gamma,r_{\alpha,\beta}-\varepsilon)-u'(\alpha,\beta,r_{\alpha,\beta}-\varepsilon)\right| \le \frac{1}{8}\left|u'(\alpha,\beta,r_{\alpha,\beta})\right| \tag{5.50}
$$

for $|\gamma-\beta|<\delta$ and our claim follows.

Now assume that B is not open. Equation (5.18) implies that there exist $\beta\in B$ and a sequence (β_j) in C such that $\beta_j\to\beta$ and $r_{\alpha,\beta_j}\to T\in[0,\infty]$. Assume first that $T>r_{\alpha,\beta}$. Then we can assume that there exists $c'\in(0,c)$ such that $r_{\alpha,\beta_j}\ge r_{\alpha,\beta}+c'$ for all j. We can also assume that ε in Step 2 is such that $0<\varepsilon<c'$. Since $u(\alpha,\beta,r_{\alpha,\beta})=0$ and $u'(\alpha,\beta,r_{\alpha,\beta})<0$, there exists $0<\varepsilon'\le\varepsilon$ such that

$$
0<u(\alpha,\beta,r_{\alpha,\beta}-\varepsilon')<\frac{1}{4}\left|u'(\alpha,\beta,r_{\alpha,\beta})\right|\varepsilon. \tag{5.51}
$$

By continuous dependence on initial data, there exists $\delta'\in(0,\delta)$ such that

$$
u(\alpha,\gamma,r_{\alpha,\beta}-\varepsilon')<2u(\alpha,\beta,r_{\alpha,\beta}-\varepsilon') \tag{5.52}
$$

when $|\gamma-\beta|<\delta'$. Now let j_0 be such that $|\beta_j-\beta|<\delta'$ for $j\geq j_0$. By Step 2, for $|r-r_{\alpha,\beta}|\leq\varepsilon$ and $j\geq j_0$ we have

$$|u'(\alpha,\beta_j,r)| = |u'(\alpha,\beta,r_{\alpha,\beta})|+u'(\alpha,\beta,r_{\alpha,\beta})-u'(\alpha,\beta_j,r)\geq\frac{1}{2}|u'(\alpha,\beta,r_{\alpha,\beta})|. \quad (5.53)$$

Therefore for $j\geq j_0$,

$$\begin{aligned}u(\alpha,\beta_j,r_{\alpha,\beta}+\varepsilon)&\leq u(\alpha,\beta_j,r_{\alpha,\beta}-\varepsilon')-\min_{|r-r_{\alpha,\beta}|\leq\varepsilon}|u'(\alpha,\beta_j,r)|(\varepsilon+\varepsilon')\\&<2u(\alpha,\beta,r_{\alpha,\beta}-\varepsilon')-\frac{1}{2}|u'(\alpha,\beta,r_{\alpha,\beta})|\varepsilon<0.\end{aligned} \quad (5.54)$$

Then we obtain a contradiction since $\beta_j\in C$. Now assume that $T\leq r_{\alpha,\beta}$. By Step 2 we have

$$|u'(\alpha,\beta_j,r_{\alpha,\beta_j})-u'(\alpha,\beta,r_{\alpha,\beta})| = |u'(\alpha,\beta,r_{\alpha,\beta})|\leq\frac{1}{2}|u'(\alpha,\beta,r_{\alpha,\beta})| \quad (5.55)$$

for $j\geq j_0$ and we get a contradiction.

Now we can complete the proof of Theorem 1.3.

(i) Let $\alpha>0$ be fixed. By Proposition 5.7, there exists a unique $\beta>0$ such that $u(\alpha,\beta,r_{\alpha,\beta})=u'(\alpha,\beta,r_{\alpha,\beta})=0$. With s and t defined in (2.1), we set

$$w(r)=\left(\frac{r_{\alpha,\beta}}{R}\right)^s u\left(\alpha,\beta,\frac{r_{\alpha,\beta}}{R}r\right),\quad z(r)=\left(\frac{r_{\alpha,\beta}}{R}\right)^t v\left(\alpha,\beta,\frac{r_{\alpha,\beta}}{R}r\right),\quad 0\leq r\leq R. \quad (5.56)$$

Then (w,z) is a nontrivial radial solution of problem (1.1).

(ii) follows from Proposition 5.7. □

Acknowledgment

The author would like to thank the referees for the useful comments and suggestions.

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