RATIONAL TORAL RANKS IN CERTAIN ALGEBRAS

YASUSUKE KOTANI and TOSHIHIRO YAMAGUCHI

Received 24 March 2004 and in revised form 17 September 2004

We calculate the rational toral ranks of two spaces whose cohomologies are isomorphic and note that rational toral rank is a rational homotopy invariant but not a cohomology invariant.


1. Introduction. Let \( \text{rk}_0(Y) \) be the rational toral rank of a simply connected space \( Y \), that is, the largest integer \( r \) such that an \( r \)-torus \( T^r = S^1 \times \cdots \times S^1 \) (\( r \)-factors) can act continuously on a CW-complex which has the rational homotopy type of \( Y \) with all its isotropy subgroups finite. For example, \( \text{rk}_0(Y) = 1 \) if \( Y \) has the rational homotopy type of an odd-dimensional sphere \( S^{2n+1} \).

Let \( \mathbb{Q} \) be the field of the rational numbers. For a finite-dimensional \( \mathbb{Q} \)-commutative graded algebra \( A^* \) with \( A^0 = \mathbb{Q} \) and \( A^1 = 0 \), we put

\[
\mathfrak{M}_{A^*} = \{ \text{rational homotopy type of } Y \mid H^*(Y; \mathbb{Q}) \cong A^* \},
\]

\[
\mathfrak{r}_{A^*} = \{ \text{rk}_0(Y) \mid H^*(Y; \mathbb{Q}) \cong A^* \},
\]

the set of rational toral ranks in \( \mathfrak{M}_{A^*} \). For example, if \( A^* = A^\text{even} \), then the Euler characteristic is nonzero, so there must be fixed points; hence, \( \mathfrak{r}_{A^*} = \{ 0 \} \).

Note that \( \mathfrak{M}_{A^*} \) and \( \mathfrak{r}_{A^*} \) are not empty sets since there exists the formal space whose cohomology is isomorphic to \( A^* \) (see below), and that \( \mathfrak{r}_{A^*} \) is at most finite even if \( \mathfrak{M}_{A^*} \) is infinite. In this paper, we calculate \( \mathfrak{r}_{A^*} \) for certain commutative graded algebras \( A^* \).

\textbf{Theorem 1.1.} For the following four algebras \( A^* \):

1. \( A^* \cong H^*(S^2 \vee S^2 \vee S^5; \mathbb{Q}) \),
2. \( A^* \cong H^*((S^3 \times S^8)\#(S^3 \times S^8); \mathbb{Q}) \),
3. \( A^* \cong H^*((S^2 \vee S^2) \times S^3; \mathbb{Q}) \),
4. \( A^* \cong H^*((S^2 \times S^5)\#(S^2 \times S^5); \mathbb{Q}) \),
the rational toral ranks in \( \mathfrak{M}_{A^*} \) are listed in Table 1.1, where \( \mathfrak{M}_{A^*} = \{ X, Y \} \) with a formal space \( X \) and a nonformal space \( Y \).

Here \( \vee \) and \( \# \) denote a one point union (wedge) and a connected sum, respectively. For these \( A^* \), we can check that \( \mathfrak{M}_{A^*} \) is two points as in [5] or [6].

What do we know about the set \( \mathfrak{r}_{A^*} \), namely, the function \( \text{rk}_0 : \mathfrak{M}_{A^*} \to \{ 0, 1, 2, \ldots \} \)? For example, We consider the following questions.

\textbf{Question 1.2.} Suppose that \( A^* \) is a Poincaré duality algebra. Then, for \( X, Y \in \mathfrak{M}_{A^*} \), is \( \text{rk}_0(X) \leq \text{rk}_0(Y) \) if \( X \) is formal?
Table 1.1. The rational toral ranks in \( M^{A\ast} \).

<table>
<thead>
<tr>
<th>Algebra</th>
<th>( \text{rk}_0(X) )</th>
<th>( \text{rk}_0(Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(2)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(3)</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(4)</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

A simply connected space \( Y \) is called (rationally) elliptic if \( \dim \pi_*(Y) \otimes \mathbb{Q} < \infty \) and \( \dim H^*(Y; \mathbb{Q}) < \infty \).

QUESTION 1.3. For \( X, Y \in M^{A\ast} \), is \( \text{rk}_0(X) \leq \text{rk}_0(Y) \) if \( Y \) is elliptic?

QUESTION 1.4. Is \( r^{A\ast} = \{a, a + 1, \ldots, b - 1, b\} \) for some integers \( a \leq b \)? Namely, are there no gaps in the sequence of integers of \( r^{A\ast} \)?

Notice that, for our examples, the answer is positive for these questions.

For the proof of Theorem 1.1, we use the Sullivan minimal model \( M(Y) \) of a simply connected space \( Y \) of finite type. It is a free \( \mathbb{Q} \)-commutative differential graded algebra (d.g.a.) \((\wedge V, d)\) with a \( \mathbb{Q} \)-graded vector space \( V = \bigoplus_{i \geq 1} V_i \), where \( \dim V_i < \infty \) and a minimal differential, that is, \( d(V_i) \subset (\wedge^+ V \cdot \wedge^+ V)^{i+1} \) and \( d \circ d = 0 \). Here \( \wedge V \) (the \( \mathbb{Q} \)-polynomial algebra over \( V^{\text{even}} \)) \( \otimes \) (the \( \mathbb{Q} \)-exterior algebra over \( V^{\text{odd}} \)) and \( \wedge^+ V \) is the ideal of \( \wedge V \) generated by elements of positive degree. Denote the degree of an element \( x \) of a graded algebra as \( |x| \). Then \( xy = (-1)^{|x||y|}yx \) and \( d(xy) = d(x)y + (-1)^{|x|}xd(y) \). Notice that \( M(Y) \) determines the rational homotopy type of \( Y \).

See [3] for a general introduction and notation: for example, for the notion of Koszul-Sullivan (KS) extension. Especially note that \( H^*(M(Y)) \cong H^*(Y; \mathbb{Q}) \) and a space \( Y \) is said to be formal if there is a d.g.a. map \( M(Y) \to (H^*(Y; \mathbb{Q}), 0) \) which induces an isomorphism of cohomologies. The formal minimal model \( M^{A\ast} \) is constructed by a free commutative resolution of the algebra \( A^{\ast} \) [5]. Throughout this paper, \( \mathbb{Q}\langle x, y, \ldots \rangle \) denotes the \( \mathbb{Q} \)-graded vector space generated by \( \{x, y, \ldots\} \).

2. Preliminaries. Let \( Y \) be a simply connected space of finite type with minimal model \( M(Y) = (\wedge V, d) \). If an \( r \)-torus \( T^r \) acts on \( Y \), there is a KS extension, with \( |t_i| = 2 \) for \( i = 1, \ldots, r \),

\[
(\mathbb{Q}[t_1, \ldots, t_r], 0) \to (\mathbb{Q}[t_1, \ldots, t_r] \otimes \wedge V, D) \to (\wedge V, d), \tag{2.1}
\]

which is induced from the Borel fibration [2]

\[
Y \to ET^r \times_{T^r} Y \to BT^r. \tag{2.2}
\]

In particular, the fact that (2.1) is a KS extension entails that, \( Dt_i = 0 \) and for \( v \in V \), \( Dv \equiv dv \) modulo the ideal \((t_1, \ldots, t_r)\), that is,

\[
Dv = dv + \sum_{i_1 + \cdots + i_r > 0} h_{i_1, \ldots, i_r} t_1^{i_1} \cdots t_r^{i_r} \tag{2.3}
\]

with \( h_{i_1, \ldots, i_r} \in \wedge V \). The differential \( D \) also satisfies \( D \circ D = 0 \).
**Lemma 2.1** [4, Proposition 4.2]. Suppose that \( \dim H^*(Y; \mathbb{Q}) < \infty \). Then, \( \text{rk}_0(Y) \geq r \) if and only if there is a KS extension (2.1) satisfying \( \dim H^*(\mathbb{Q}[t_1, \ldots, t_r] \otimes \wedge V, D) < \infty \).

So we may try to construct inductively for \( 1, \ldots, i \), the KS extensions:

\[
(Q[t_i], 0) \to (Q[t_1, \ldots, t_i] \otimes \wedge V, D_i) \to (Q[t_1, \ldots, t_{i-1}] \otimes \wedge V, D_{i-1})
\]

(2.4)

satisfying \( \dim H^*(\mathbb{Q}[t_1, \ldots, t_i] \otimes \wedge V, D) < \infty \) in general. In the following, we consider the particular case of \( i = 1 \).

**Lemma 2.2.** Suppose that \( H^{n+2}(\wedge V, d) = 0 \) and \( H^n(\mathbb{Q}[t] \otimes \wedge V, D) = \mathbb{Q}(y_1, \ldots, y_m) \). Then, \( H^{n+2}(\mathbb{Q}[t] \otimes \wedge V, D) \subset \mathbb{Q}(y_1t, \ldots, y_mt) \). Moreover, if \( H^{n+1}(\wedge V, d) = 0 \), then the inclusion is an equality.

**Proof.** Let \( \alpha + \alpha' t \) be a \( D \)-cocycle in \( (\mathbb{Q}[t] \otimes \wedge V)^{n+2} \) with \( \alpha \in (\wedge V)^{n+2} \) and \( \alpha' \in (\mathbb{Q}[t] \otimes \wedge V)^n \). Then we have \( D\alpha = -D(\alpha')t \), and consequently, \( d\alpha = 0 \).

Since \( H^{n+2}(\wedge V, d) = 0 \), there is an element \( \beta \in (\wedge V)^{n+1} \) such that \( D\beta = \alpha \). Let \( D\beta = \alpha + \alpha'' t \) for some \( \alpha'' \in (\mathbb{Q}[t] \otimes \wedge V)^n \). Then, since

\[
0 = D^2\beta = D\alpha + D(\alpha'')t = -D(\alpha' - \alpha'')t,
\]

(2.5)

we see that \( \alpha' - \alpha'' \) is a \( D \)-cocycle in \( (\mathbb{Q}[t] \otimes \wedge V)^n \).

Since \( H^n(\mathbb{Q}[t] \otimes \wedge V, D) = \mathbb{Q}(y_1, \ldots, y_m) \), we can denote \( \alpha' - \alpha'' = c_1y_1 + \cdots + c_my_m + D\beta' \) for some \( c_1, \ldots, c_m \in \mathbb{Q} \) and \( \beta' \in (\mathbb{Q}[t] \otimes \wedge V)^{n-1} \). Then we have

\[
\alpha + \alpha' t = \alpha + (\alpha'' + c_1y_1 + \cdots + c_my_m + D\beta')t = c_1y_1t + \cdots + c_my_mt + D(\beta + \beta')t.
\]

(2.6)

Hence \( [\alpha + \alpha' t] = [c_1y_1t + \cdots + c_my_mt] \) in \( H^{n+2}(\mathbb{Q}[t] \otimes \wedge V, D) \). Thus we have \( H^{n+2}(\mathbb{Q}[t] \otimes \wedge V, D) \subset \mathbb{Q}(y_1t, \ldots, y_mt) \).

Suppose that \( c_1y_1t + \cdots + c_my_mt = D(\eta + \eta' t) \) for some \( \eta \in (\wedge V)^{n+1} \) and \( \eta' \in (\mathbb{Q}[t] \otimes \wedge V)^{n-1} \). Then we have \( d\eta = 0 \) since \( d\eta \notin \text{Ideal}(t) \). If \( H^{n+1}(\wedge V, d) = 0 \), there is an element \( \theta \in (\wedge V)^n \) such that \( D\theta = \eta \). Let \( D\theta = \eta + \eta'' t \) for some \( \eta'' \in (\mathbb{Q}[t] \otimes \wedge V)^{n-1} \). Then we have

\[
(c_1y_1 + \cdots + c_my_m)t = D(\eta + \eta' t) = D(D\theta - \eta'' t + \eta' t) = D(\eta' - \eta'')t.
\]

(2.7)

However, \( c_1y_1 + \cdots + c_my_m \notin \text{Im} D \) unless \( c_1 = \cdots = c_m = 0 \). Thus, if \( H^{n+1}(\wedge V, d) = 0 \), \( y_1t, \ldots, y_mt \) are linearly independent in \( H^{n+2}(\mathbb{Q}[t] \otimes \wedge V, D) \). \( \square \)
A commutative graded algebra \( A^* \) with \( \dim A^* < \infty \) will be said to have formal dimension \( n \) if \( A^n \neq 0 \) and \( A^i = 0 \) for all \( i > n \). For example, the formal dimensions of \((1), (2), (3), \) and \((4)\) are 5, 11, 5, and 7, respectively.

**Lemma 2.3** [4, Lemma 5.4]. Suppose that \( H^\ast(\wedge V, d) \) and \( H^\ast(\mathbb{Q}[t] \otimes \wedge V, D) \) have formal dimensions \( n \) and \( n' \), respectively. Then \( n' = n - 1 \). If one algebra satisfies Poincaré duality, so does the other.

From **Lemma 2.1** the following corollary may be useful to estimate a rational toral rank to be nonzero.

**Corollary 2.4.** Suppose that \( H^\ast(\wedge V, d) \) has formal dimension \( n \). Then, \( \dim H^\ast(\mathbb{Q}[t] \otimes \wedge V, D) < \infty \) if and only if \( H^n(\mathbb{Q}[t] \otimes \wedge V, D) = H^{n+1}(\mathbb{Q}[t] \otimes \wedge V, D) = 0. \)

**Proof.** The “if” part is proved as follows. Since \( H^{n+2i}(\wedge V, d) = 0 \) for \( i > 0 \), we have \( H^{n+2i}(\mathbb{Q}[t] \otimes \wedge V, D) = 0 \) for \( i \geq 0 \) from Lemma 2.2. Similarly, since \( H^{n+2i-1}(\wedge V, d) = 0 \) for \( i > 0 \), we have \( H^{n+2i-1}(\mathbb{Q}[t] \otimes \wedge V, D) = 0 \) for \( i > 0 \) from Lemma 2.2. Hence we have \( H^{n+i}(\mathbb{Q}[t] \otimes \wedge V, D) = 0 \) for \( i \geq 0 \), that is, \( \dim H^\ast(\mathbb{Q}[t] \otimes \wedge V, D) < \infty. \)

The “only if” part follows from **Lemma 2.3.**

**Proposition 2.5.** Suppose that \( H^\ast(\wedge V, d) \) has formal dimension \( n \) and \((\wedge Z, D)\) is a minimal d.g.a. Then \( H^\ast(\wedge Z, D) \) has formal dimension \( n - 1 \) and \( Z^{< n} = \mathbb{Q}(t) \oplus V^{< n} \) with \( D \equiv d \mod(t) \) on \( V^{< n} \) if and only if \( Z = \mathbb{Q}(t) \oplus V \) and \( D \equiv d \mod(t) \), that is, there is a KS extension

\[
(\mathbb{Q}[t], 0) \rightarrow (\wedge Z, D) = (\mathbb{Q}[t] \otimes \wedge V, D) \rightarrow (\wedge V, d)
\]  

such that \( \dim H^\ast(\mathbb{Q}[t] \otimes \wedge V, D) < \infty. \)

**Proof.** The “if” part is obvious from **Lemma 2.3.**

Now we show the “only if” part. For some \( k \geq n \), assume that \( Z^{< k} = \mathbb{Q}(t) \oplus V^{< k} \) with \( Dv \equiv dv \mod(t) \) for \( v \in V^{< k} \). Then an element in \( H^{k+2}(\wedge Z^{< k}, D) \) can be written using \([\alpha + \alpha' t] \) with \( \alpha \in (\wedge Z^{< k})^{k+2} \) and \( \alpha' \in (\wedge Z^{< k})^{k} \). Since \( D(\alpha + \alpha' t) = 0 \), we have \( d\alpha = 0 \). Now we give a map

\[
\rho_{k+1} : H^{k+2}(\wedge Z^{< k}, D) \rightarrow H^{k+2}(\wedge V^{< k}, d)
\]

where \( \rho_{k+1}([\alpha + \alpha' t]) = [\alpha] \). It is well defined. Indeed, if \([\alpha_1 + \alpha'_1 t] = [\alpha_2 + \alpha'_2 t] \) in \( H^{k+2}(\wedge Z^{< k}, D) \), then \( \alpha_1 + \alpha'_1 t = \alpha_2 + \alpha'_2 t + D(\beta + \beta' t) \) for some \( \beta \in (\wedge Z^{< k})^{k+1} \) and \( \beta' \in (\wedge Z^{< k})^{k-1} \). Let \( D\beta = d\beta + \beta'' t \). Then we have

\[
(\alpha_1 - \alpha_2) + (\alpha'_1 - \alpha'_2) t = d\beta + (\beta'' + D(\beta')) t.
\]

So \( \alpha_1 - \alpha_2 = d\beta \). Hence \([\alpha_1] = [\alpha_2] \) in \( H^{k+2}(\wedge V^{< k}, d) \).

Since \( \rho_{k+1} \) is bijective, from the following paragraphs we see that \( Z^{< k+1} = V^{< k+1} \) with \( Dv \equiv dv \mod(t) \) for \( v \in V^{< k+1} \) from the construction of minimal d.g.a.’s such that \( H^{k+2}(\wedge Z, D) = H^{k+2}(\wedge V, d) = 0 \). Thus we have inductively \( Z = \mathbb{Q}(t) \oplus V \) with \( Dv \equiv dv \mod(t) \) for \( v \in V \).
Now we show that \( \rho_{k+1} \) is injective. Suppose that \( \rho_{k+1}([\alpha + \alpha']t) = [\alpha] = 0 \). Then there is an element \( \beta \in \langle \wedge V^{\leq k} \rangle^{k+1} \) such that \( d\beta = \alpha \). Let \( D\beta = \alpha + \alpha''t \). Since \( D(\alpha + \alpha't) = 0 \) and \( D(\alpha + \alpha''t) = D^2(t) = 0 \), we have \( D(\alpha' - \alpha'') = 0 \). Since \( H^k(\wedge Z^{\leq k}, D) = 0 \), \( \alpha' - \alpha'' = D\beta' \) for some \( \beta' \in \langle \wedge Z^{\leq k} \rangle^{k-1} \). Then we have

\[
\alpha + \alpha' t = \alpha + (\alpha'' + D\beta')t = D(\beta + \beta' t).
\] (2.11)

Hence \( [\alpha + \alpha' t] = 0 \).

Now we show that \( \rho_{k+1} \) is surjective. Let \( [\alpha] \in H^{k+2}(\wedge V^{\leq k}, d) \). Since \( d\alpha = 0 \), we can denote \( D\alpha = yt \) with \( y \in (\wedge Z^{\leq k})^{k+1} \). Since \( H^{k+1}(\wedge Z^{\leq k}, D) = 0 \), \( y = D\eta \) for some \( \eta \in (\wedge Z^{\leq k})^k \). Then we have

\[
D(\alpha - \eta t) = D\alpha - D(\eta)t = yt - yt = 0.
\] (2.12)

Hence there is an element \( [\alpha - \eta t] \in H^{k+2}(\wedge Z^{\leq k}, d) \) such that \( f([\alpha - \eta t]) = [\alpha] \). \( \square \)

From Lemma 2.1, we have the following.

**Corollary 2.6.** Let \( M(Y) = (\wedge V, d) \) with cohomology of formal dimension \( n \). If there is a minimal d.g.a. \( (\wedge V, D) \) such that \( H^*(\wedge V, D) \) has formal dimension \( n - 1 \) and \( Z^{\leq n} = \mathbb{Q}(t) \oplus V^{\leq n} \) with \( D \equiv d \mod(t) \) on \( V^{\leq n} \), then \( M(ES^1 \times S^1 Y) \cong (\wedge V, D) \), that is, \( \operatorname{rk}_0(Y) \geq 1 \).

In the following, \( X \) is formal and \( Y \) is nonformal.

### 3. Examples

**Example 3.1.** Let \( X = S^2 \vee S^2 \vee S^5 \). Then \( \chi_H(X) = \sum_i (-1)^i \dim H^i(X; \mathbb{Q}) = 2 > 0 \). Recall

\[
\chi_H(ES^1 \times S^1 X) = \chi_H(X) \cdot \chi_H(BS^1)
\] (3.1)

for a Borel fibration \( X \to ES^1 \times S^1 X \to BS^1 \). Since \( \chi_H(BS^1) = \infty \) we have \( \chi_H(ES^1 \times S^1 X) = \infty \), that is, \( \dim H^*(ES^1 \times S^1 X; \mathbb{Q}) = \infty \). From Lemma 2.1, \( \operatorname{rk}_0(X) = 0 \). By the same argument, we have \( \operatorname{rk}_0(Y) = 0 \).

Note that \( \chi_H(X) = \chi_H(Y) = 0 \) in (2), (3), and (4).

**Remark 3.2.** Even if \( X \) is a wedge of spaces, \( \operatorname{rk}_0(X) \) may not be zero. For example, \( M(S^3 \vee S^2 \vee S^4) = (\wedge V, d) = (\wedge (x, y, z, \ldots), d) \) with \( |x| = |y| = 3 \) and \( |z| = 4 \) and \( dx = dy = dz = 0 \). On the other hand, \( M(S^2 \vee S^3)^{\leq 4} = (\wedge Z, D)^{\leq 4} = (\wedge (t, x, y, z), D) \) with \( |t| = 2, Dt = Dx = 0, Dy = t^2, \) and \( Dz = xt \). From Corollary 2.6, we have \( \operatorname{rk}_0(S^3 \vee S^3 \vee S^4) \geq 1 \).

**Example 3.3.** Let \( X = (S^3 \times S^8)\#(S^3 \times S^8) \). Then

\[
A^+ = H^*(X; \mathbb{Q}) = \frac{\wedge (x, y) \otimes \mathbb{Q}[w, u]}{(xy, xu, xu - yu, yw, w^2, wu, u^2)}
\] (3.2)

with \( |x| = |y| = 3, |w| = |u| = 8 \) and \( X \) has the minimal model

\[
(\wedge V_X, d) = (\wedge (x, y, w, u, v_1, v_2, v_3, v_4, v_5, v_6, v_7, z_1, \ldots), d)
\] (3.3)
with $|v_1|=5$, $|v_2|=|v_3|=|v_4|=10$, $|v_5|=|v_6|=|v_7|=15$, $|z_1|=7$ and $dx=dy=dw=du=0$, $dv_1=x\gamma$, $dv_2=xu$, $dv_3=xw-yu$, $dv_4=yw$, $dv_5=w^2$, $dv_6=wu$, $dv_7=u^2$, $dz_1=xv_1$, ...

From $D\circ D=0$, we have $Dx=Dy=0$, $Du=\lambda xt^3$, and $Dw=-\lambda yt^3$ for $\lambda \in Q$. Assume $\dim H^*(Q[t]\otimes V_X,D)<\infty$. From Lemma 2.3, $\lambda \neq 0$. Let $Dv_1=x\gamma+at^3$ for $a \in Q$ and $Dz_1=xv_1+ht$ for $h \in (Q[t]\otimes V_X,D)^{\phi}$. Then $0=D^2z_1=-axt^3+D(h)t$.

But there is no element $h$ such that $Dh=axt^2$. Hence we have $a=0$. Since $H^*(X;Q)$ satisfies Poincaré duality with formal dimension 11, so does $H^*(Q[t]\otimes V_X,D)$ with formal dimension 10 from Lemma 2.3. Since $H^3(Q[\gamma]\otimes V_X,D)=Q(x,\gamma)$ and $H^4(\otimes V_X,d)=0$ for $4\leq i \leq 7$, we have $H^7(Q[t]\otimes V_X,D)=Q(x^2,\gamma t^2)$ from Lemma 2.2. But

$$x\cdot xt^2=x\cdot yt^2=0 \quad (3.4)$$

in $H^{10}(Q[t]\otimes V_X,D)$ since $a=0$. This contradicts Poincaré duality. Thus $\dim H^*(Q[t]\otimes V_X,D)=\infty$. From Lemma 2.1, we have $rk_0(X)=0$.

Let $M(Y)=(\otimes V_Y,d)=((\otimes (x,\gamma,z),d)$ with $|x|=|y|=3$, $|z|=5$ and $dx=dy=0$, $dz=x\gamma$. Then $H^*(Y;Q)\equiv A^*$. Put $Dx=Dy=0$ and $Dz=x\gamma+t^3$. Then $\dim H^*(Q[t]\otimes V_Y,D)<\infty$. From Lemma 2.1, we have $rk_0(Y)\geq 1$. Also, for any $D$, we have $Dx=Dy=0$. Thus $H^*(Q[t_1,t_2]\otimes V_Y,D)=\infty$. From the case of $r=2$ in Lemma 2.1, we have $rk_0(Y)=1$.

**Example 3.4.** Let $X=(S^2\vee S^2)^{3}$. Then $A^*=H^*(X;Q)=Q[x_1,x_2]\otimes (y)/(x_1^2,x_1x_2, x_2^2)$ with $|x_1|=2$, $|y|=3$. When $D=d$, except for $Dy=t^2$, $(Q[t]\otimes V_X,D)$ is the minimal model of $(S^2\vee S^2)^{3}$. Hence $rk_0(X)\geq 1$. In general, if $Dy=0$, $[x_1\gamma] \neq 0 \in H^3(Q[t]\otimes V_X,D)$, then $\dim H^*(Q[t]\otimes V_X,D)=\infty$ from Lemma 2.2. If $Dy \neq 0$, $H^{odd}(Q[t]\otimes V_X,D)=0$ from Lemma 2.3. In each case, $\dim H^*(Q[t_1,t_2]\otimes V_X,D)$ cannot be finite. From the case of $r=2$ in Lemma 2.1, we have $rk_0(X)=1$.

Let $Y$ be the nonformal space with $H^*(Y;Q)\equiv A^*$. Then $M(Y)=(\otimes V_Y,d)$ is given by

$$V_Y^{\leq 5}=Q\langle x_1,x_2,y,z_1,z_2,z_3,u_1,u_2,v_1,v_2,v_3 \rangle \quad (3.5)$$

with $|x_1|=2$, $|y|=|z_1|=3$, $|u_1|=4$, $|v_1|=5$ and $dx_1=dx_2=dy=0$, $dz_1=x_1^2$, $dz_2=x_1x_2$, $dz_3=x_2^2$, $du_1=x_1z_2-x_2z_1$, $du_2=x_1z_3-x_2z_2-x_2y$, $dv_1=x_1u_1-z_1z_2$, $dv_2=x_1u_2-x_2u_1-z_1z_3+z_2y$, $dv_3=x_2u_2-z_2z_3+z_3y$. Here $H^5(\otimes V_Y,d)=Q(x_1y,xy^2)$.

Now we show that $t^3 \neq 0$ in $H^6(Q[t]\otimes V_Y,D)$. Let $Dx_1=dx_2=0$, $Dy=ax_1t+bx_2t+ct^2$ for $a,b,c \in Q$ and $Dz_1=dz_1+ai_1x_1t+b_1x_2t+c_1t^2$ for $a_i,b_i,c_i \in Q$. Assume that $t^3=D(px_1y+qxy+y\gamma t+fz_1t+gz_2t+hz_3t)$ for some $p,q,e,f,g,h \in Q$. Since the right-hand side is equal to

$$(pa+f)x_1t^2+(pb+qa+g)x_1x_2t+(q\gamma+h)x_2^2t$$
$$+(pc+ea+fa+ga+ha_3)x_1t^2+(qc+eb+fb_1+gb_2+hb_3)x_2t^2$$
$$+(ec+fc_1+gc_2+hc_3)t^3, \quad (3.6)$$

...
we have
\[ pc + ea - paa_1 - pba_2 - qaa_2 - qba_3 = 0, \]
\[ qc + eb - pab_1 - pbb_2 - qab_2 - qbb_3 = 0, \]  
\[ ec - pac_1 - pbc_2 - qac_2 - qbc_3 = 1. \]  
(3.7)

On the other hand, let \( Du_i = du_i + e_i y t + f_i z_1 t + g_i z_2 t + h_i z_3 t \) for \( e_i, f_i, g_i, h_i \in \mathbb{Q} \) and \( Dv_i = dv_i + l_i u_1 t + m_i u_2 t \) for \( l_i, m_i \in \mathbb{Q} \). Since

\[ 0 = D^2 u_1 \]
\[ = (a_1 + f_1) x_1^2 t + (b_1 - a_1 + g_1) x_1 x_2 t + (-b_1 + h_1) x_2^2 t \]
\[ + (c_1 + e_1 a + f_1 a_1 + g_1 a_2 + h_1 a_3) x_1 t^2 \]
\[ + (-c_1 + e_1 b + f_1 b_1 + g_1 b_2 + h_1 b_3) x_2 t^2 \]
\[ + (e_1 c + f_1 c_1 + g_1 c_2 + h_1 c_3) t^3, \]

\[ 0 = D^2 u_2 \]
\[ = (a_2 + f_2) x_1^2 t + (b_2 - a_2 - a + g_2) x_1 x_2 t + (-b_2 - b + h_2) x_2^2 t \]
\[ + (c_2 + e_2 a + f_2 a_1 + g_2 a_2 + h_2 a_3) x_1 t^2 \]
\[ + (-c_2 - c + e_2 b + f_2 b_1 + g_2 b_2 + h_2 b_3) x_2 t^2 \]
\[ + (e_2 c + f_2 c_1 + g_2 c_2 + h_2 c_3) t^3, \]

\[ 0 = D^2 v_1 \]
\[ = e_1 x_1 y t + (f_1 + a_2) x_1 z_1 t + (g_1 - a_1 + l_1) x_1 z_2 t + (h_1 + m_1) x_1 z_3 t \]
\[ - m_1 x_2 y t + (b_1 - l_1) x_2 z_1 t + (-b_1 - m_1) x_2 z_2 t \]
\[ + (l_1 e_1 + m_1 e_2) y t^2 + (c_2 + l_1 f_1 + m_1 f_2) z_1 t^2 \]
\[ + (-c_1 + l_1 g_1 + m_1 g_2) z_2 t^2 + (l_1 h_1 + m_1 h_2) z_3 t^2, \]  
(3.8)

\[ 0 = D^2 v_2 \]
\[ = (e_2 + a_2) x_1 y t + (f_2 + a_3) x_1 z_1 t \]
\[ + (g_2 - a + l_2) x_1 z_2 t + (h_2 - a_1 + m_2) x_1 z_3 t \]
\[ + (e_1 + b_2 - m_2) x_2 y t + (f_1 + b_3 - l_2) x_2 z_1 t \]
\[ + (g_1 - b - m_2) x_2 z_2 t + (h_1 - b_1) x_2 z_3 t \]
\[ + (c_2 + l_2 e_1 + m_2 e_2) y t^2 + (c_3 + l_2 f_1 + m_2 f_2) z_1 t^2 \]
\[ + (-c_2 + l_2 g_1 + m_2 g_2) z_2 t^2 + (-c_1 + l_3 h_1 + m_2 h_2) z_3 t^2, \]

\[ 0 = D^2 v_3 \]
\[ = a_3 x_1 y t + (a_3 + l_3) x_1 z_2 t + (-a_2 - a + m_3) x_1 z_3 t \]
\[ + (e_2 + b_3 - m_3) x_2 y t + (f_2 - l_3) x_2 z_1 t \]
\[ + (g_2 + b_3 - m_3) x_2 z_2 t + (h_2 - b_2 - b) x_2 z_3 t \]
\[ + (c_3 + l_3 e_1 + m_3 e_2) y t^2 + (l_3 f_1 + m_3 f_2) z_1 t^2 \]
\[ + (c_3 + l_3 g_1 + m_3 g_2) z_2 t^2 + (-c_2 - c + l_3 h_1 + m_3 h_2) z_3 t^2, \]
we have

\[
\begin{align*}
a &= -2a_2 + b_3, & b &= a_1 - 2b_2, & c &= -a_1a_2 + a_1b_3 - b_2b_3, \\
a_3 &= b_1 = 0, & c_1 &= (a_1 - b_2)b_2, & c_2 &= a_2b_2, & c_3 &= -(a_2 - b_3)a_2.
\end{align*}
\]  

(3.9)

Hence (3.7) will be

\[
\begin{align*}
( -2a_2 + b_3)(e - pb_2 - qa_2) &= 0, \\
( a_1 - 2b_2)(e - pb_2 - qa_2) &= 0, \\
( -a_1a_2 + a_1b_3 - b_2b_3)(e - pb_2 - qa_2) &= 1,
\end{align*}
\]

(3.10) (3.11) (3.12)

respectively. By (3.12), \(e - pb_2 - qa_2 \neq 0\) and \(-a_1a_2 + a_1b_3 - b_2b_3 \neq 0\). Then, by (3.10) and (3.11), \(b_3 = 2a_2\) and \(a_1 = 2b_2\), respectively. But this contradicts \(-a_1a_2 + a_1b_3 - b_2b_3 \neq 0\). Thus \(t^3 = 0\) in \(H^0(\mathbb{Q}[t] \otimes V_Y, D)\).

Since \(H^*(\wedge V_Y, d)\) has formal dimension 5, from Lemma 2.3, we have \(\dim H^*(\mathbb{Q}[t] \otimes \wedge V_Y, D) = \infty\). From Lemma 2.1, we have \(\operatorname{rk}_0(Y) = 0\).

**Example 3.5.** Let \(X = (S^2 \times S^5) \# (S^2 \times S^5)\). Then

\[
A^* = H^*(X; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, x_2] \otimes \wedge (y_1, y_2)}{(x_1^2, x_1x_2, x_2^2, x_1y_1 - x_2y_2, x_1y_2, x_2y_1, y_1y_2)}
\]

(3.13)

with \(|x_i| = 2, |y_i| = 5\) and \(X\) has a minimal model \(M(X) = M_{A^*} = (\wedge V_X, d)\) where

\[
V_X^{S^7} = \mathbb{Q} \langle x_1, x_2, z_1, z_2, z_3, u_1, u_2, y_1, y_2, v_1, v_2, v_3, w_1, \ldots, w_9, s_1, \ldots, s_{18} \rangle
\]

(3.14)

with \(|x_i| = 2, |z_i| = 3, |u_i| = 4, |y_i| = |v_i| = 5, |w_i| = 6, |s_i| = 7\) and

\[
\begin{align*}
dx_1 &= dx_2 = dy_1 = dy_2 = 0, \\
dz_1 &= x_1^2, & dz_2 &= x_1x_2, & dz_3 &= x_2^2, \\
du_1 &= x_1z_2 - x_2z_1, & du_2 &= x_1z_3 - x_2z_2, \\
dv_1 &= x_1u_1 - z_1z_2, & dv_2 &= x_1u_2 + x_2u_1 - z_1z_3, & dv_3 &= x_2u_2 - z_2z_3, \\
dw_1 &= x_1y_1 - x_2y_2, & dw_2 &= x_1y_2, & dw_3 &= x_2y_1, \\
dw_4 &= x_1v_1 - z_1u_1, & dw_5 &= x_1v_2 - z_1u_2 - z_2u_1, & dw_6 &= x_1v_3 - z_2u_2, \\
dw_7 &= x_2v_1 - z_2u_1, & dw_8 &= x_2v_2 - z_2u_2 - z_3u_1, & dw_9 &= x_2v_3 - z_3u_2, \\
ds_1 &= x_1w_1 - z_1y_1 + z_2y_2, & ds_2 &= x_1w_2 - z_1y_2, & ds_3 &= x_1w_3 - z_2y_1, \\
ds_4 &= x_1w_4 - z_1v_1, & ds_5 &= x_1w_5 - z_1v_2 + \frac{1}{2}u_1^2,
\end{align*}
\]
\begin{align*}
\text{ds}_6 &= x_1w_6 + x_1w_8 - z_1v_3 - z_2v_2 + u_1u_2, \\
\text{ds}_7 &= x_1w_7 - x_2w_4 + \frac{1}{2}u_1^2, \\
\text{ds}_8 &= x_1w_8 - x_2w_5 + u_1u_2, \\
\text{ds}_9 &= x_1w_9 - x_2w_6 + \frac{1}{2}u_2^2, \\
\text{ds}_{10} &= x_2w_1 - z_2y_1 + z_3y_2, \\
\text{ds}_{11} &= x_2w_2 - z_2y_2, \\
\text{ds}_{12} &= x_2w_3 - z_3y_1, \\
\text{ds}_{13} &= x_2w_4 - z_2v_1 - \frac{1}{2}u_1^2, \\
\text{ds}_{14} &= x_2w_5 + x_2w_7 - z_2v_2 - z_3v_1 - u_1u_2, \\
\text{ds}_{15} &= x_2w_6 - z_2v_3, \\
\text{ds}_{16} &= x_2w_7 - x_1w_6 + z_1v_3 - z_3v_1 - u_1u_2, \\
\text{ds}_{17} &= x_2w_8 - z_3v_2 - \frac{1}{2}u_2^2, \\
\text{ds}_{18} &= x_2w_9 - z_3v_3.
\end{align*}

(3.15)

Let \((\land Z, D)\) be the formal minimal model \(M_{B^*}\) for the Poincaré duality algebra

\[
B^* = \frac{\mathbb{Q}[t, x_1, x_2]}{(x_1 t^2, x_2 t^2, x_1^2 + x_2 t, x_1 x_2 - t^2, x_2^2 + x_1 t)}
\]

(3.16)

with \(|t| = |x_i| = 2\). Note \(B^*\) has formal dimension 6. Then

\[
Z^{\le 7} = \mathbb{Q}(t) \oplus V_X^{\le 7}
\]

(3.17)

with

\[
\begin{align*}
\text{Dt} &= Dx_1 = Dx_2 = 0, \\
\text{Dy}_1 &= x_2 t^2, \\
\text{Dz}_1 &= dz_1 + x_2 t, \\
\text{Dz}_2 &= dz_2 - t^2, \\
\text{Dz}_3 &= dz_3 + x_1 t, \\
\text{Du}_1 &= du_1 + z_3 t, \\
\text{Du}_2 &= du_2 - z_1 t, \\
\text{Dv}_1 &= dv_1 - u_2 t, \\
\text{Dv}_2 &= dv_2, \\
\text{Dv}_3 &= dv_3 - u_1 t, \\
\text{Dw}_1 &= dw_1, \\
\text{Dw}_2 &= dw_2 + y_1 t - z_1 t^2, \\
\text{Dw}_3 &= dw_3 + y_2 t - z_3 t^2, \\
\text{Dw}_4 &= dw_4 + v_2 t, \\
\text{Dw}_5 &= dw_5 + v_3 t, \\
\text{Dw}_6 &= dw_6 + v_1 t, \\
\text{Dw}_7 &= dw_7 + v_2 t, \\
\text{Dw}_8 &= dw_8 + v_3 t, \\
\text{Dw}_9 &= dw_9 + v_1 t, \\
\text{Ds}_1 &= ds_1 + w_3 t + u_1 t^2, \\
\text{Ds}_2 &= ds_2 - w_1 t, \\
\text{Ds}_3 &= ds_3 - w_2 t + u_2 t^2, \\
\text{Ds}_4 &= ds_4 - w_5 t + w_7 t, \\
\text{Ds}_5 &= ds_5 - w_6 t + w_8 t, \\
\text{Ds}_6 &= ds_6 - 2w_4 t + w_9 t, \\
\text{Ds}_7 &= ds_7 - w_6 t + w_8 t, \\
\text{Ds}_8 &= ds_8 - w_4 t + w_9 t, \\
\text{Ds}_9 &= ds_9 - w_5 t + w_7 t, \\
\text{Ds}_{10} &= ds_{10} - w_2 t + u_2 t^2, \\
\text{Ds}_{11} &= ds_{11} - w_3 t - u_1 t^2, \\
\text{Ds}_{12} &= ds_{12} + w_1 t, \\
\text{Ds}_{13} &= ds_{13} - w_8 t, \\
\text{Ds}_{14} &= ds_{14} + w_4 t - 2w_9 t, \\
\text{Ds}_{15} &= ds_{15} - w_7 t, \\
\text{Ds}_{16} &= ds_{16} + 2w_4 t - 2w_9 t, \\
\text{Ds}_{17} &= ds_{17} + w_7 t - w_7 t, \\
\text{Ds}_{18} &= ds_{18} + w_6 t - w_8 t.
\end{align*}
\]

(3.18)

that is, \(D \equiv d \mod (t)\) on \(V_X^{\le 7}\). From Corollary 2.6, we have \(\text{rk}_0(X) \ge 1\). Also for any \(D\) satisfying \(\dim H^*(\mathbb{Q}[t] \otimes \wedge V_X, D) < \infty\), we see \(H^{\text{odd}}(\mathbb{Q}[t] \otimes \wedge V_X, D) = 0\) from Lemma 2.3. From the case of \(r = 2\) in Lemma 2.1, we have \(\text{rk}_0(X) = 1\).

Let \(M(Y) = (\wedge V_Y, d) = (\wedge(x_1, x_2, z_1, z_2, z_3), d)\) with \(|x_1| = 2, |z_i| = 3\) and \(dx_1 = dx_2 = 0, dz_1 = x_1^2, dz_2 = x_1x_2, dz_3 = x_2^2\). Then \(H^*(Y; \mathbb{Q}) \cong A^*\).
Put \( D = d \) except for \( Dz_2 = x_1x_2 - t^2 \). Then we have \( \dim H^*(\mathbb{Q}[t] \otimes \wedge V_Y, D) < \infty \).
From the case of \( r = 1 \) in Lemma 2.1, \( \text{rk}_0(Y) \geq 1 \). From [1], we have \( \text{rk}_0(Y) = 1 \). Indeed,

\[
\text{rk}_0(Y) \leq -\chi_\pi(Y) = -\sum_i (-1)^i \dim \pi_i(Y) \otimes \mathbb{Q} = \dim V_Y^{\text{odd}} - \dim V_Y^{\text{even}} = 1.
\]

(3.19)

**References**


Yasusuke Kotani: Department of Mathematics, Faculty of Science, Kochi University, Kochi 780-8520, Japan

E-mail address: kotani@math.kochi-u.ac.jp

Toshihiro Yamaguchi: Department of Mathematics Education, Faculty of Education, Kochi University, Kochi 780-8520, Japan

E-mail address: tyamag@cc.kochi-u.ac.jp
Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers
The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal’s Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th></th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>May 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>August 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>November 1, 2009</td>
</tr>
</tbody>
</table>

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O’Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie