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ON THE GENUS OF FREE LOOP FIBRATIONS OVER F_0-SPACES

TOSHIHIRO YAMAGUCHI

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We give a lower bound of the genus of the fibration of free loops on an elliptic space whose rational cohomology is concentrated in even degrees.

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1. Introduction. In this note, all spaces are supposed to be connected and having the rational homotopy type of a CW complex of finite type. The LS category, $\operatorname{cat}(X)$, of a space X is the least integer n such that X can be covered by $n+1$ open subsets, each contractible in X. The genus, $\operatorname{genus}(\eta)$ or $\operatorname{genus}(p)$, of a fibration $\eta : F \to E \xrightarrow{p} B$ is the least integer n such that B can be covered by $n+1$ open subsets, over each of which p is a trivial fibration, in the sense of fiber homotopy type. The sectional category, $\operatorname{secat}(\eta)$, is the least integer n such that B can be covered by $n+1$ open subsets, over each of which p has a section. Let

$$\mathcal{L}_X : \Omega X \longrightarrow LX \longrightarrow X \tag{1.1}$$

be the fibration of free loops on a 2-connected space X and let $\mathcal{P}_X : \Omega X \to PX \to X$ be the path fibration. It is known that $\mathcal{L}_X$ is an interesting object in topology and geometry [1, 9]. We know that $\operatorname{cat}(X) = \operatorname{secat}(\mathcal{P}_X) = \operatorname{genus}(\mathcal{P}_X)$ (see [4, page 599]). On the other hand, since $\mathcal{L}_X$ has a section, $\operatorname{secat}(\mathcal{L}_X) = 0$. But it seems hard to know $\operatorname{genus}(\mathcal{L}_X)$ in general. In this note, we consider a certain case for X by using the argument of the Sullivan minimal model in [4].

A simply connected space is said to be elliptic if the dimensions of rational cohomology and homotopy are finite. An elliptic space X is said to be an F_0-space if the rational cohomology is concentrated in even degrees. Then there is an isomorphism $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_n]/(f_1,\dots,f_n)$ with a regular sequence $f_1,\dots,f_n$. For example, the homogeneous space G/H where G and H have same rank is an F_0-space. Note that there is a conjecture of Halperin for an F_0-space (see [3, page 516], [7]).

Theorem 1.1. *Let X be a 2-connected F_0-space of n variables. Then* $\operatorname{genus}(\mathcal{L}_X) \geq n$.

In the following, Section 2 is a preliminary in Sullivan minimal models and we prove the theorem in Section 3. Refer to [3] for the rational model theory.

2. Sullivan model of classifying map. Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model [3, Section 12] of a 2-connected space X, in which $V = \bigoplus_{i>2} V^i$ as a graded vector space. Let $\overline{V}^i = V^{i+1}$ and let $\beta : \Lambda\overline{V} \otimes \Lambda V \to \Lambda\overline{V} \otimes \Lambda V$ be the derivation ($\beta(xy) = \beta(x)y + (-1)^{\deg x} x\beta(y)$) of degree -1 with the properties $\beta(v) = \overline{v}$ and $\beta(\overline{v}) = 0$.

Then $M(\Omega X) = (\Lambda\overline{V}, 0)$ and $M(LX) \cong (\Lambda\overline{V} \otimes \Lambda V, \delta)$ with $\delta v = dv$ and $\delta\overline{v} = -\beta dv = \sum_j \pm_j \partial dv/\partial v_j \cdot \overline{v}_j$ for a basis v_j of V [9].

Let Y be a simply connected space and let $\mathrm{Der}_i M(Y)$ be the set of derivations of $M(Y)$ decreasing the degree by $i > 0$. We denote $\bigoplus_{i>0} \mathrm{Der}_i M(Y)$ by $\mathrm{Der}\, M(Y)$. The Lie bracket is defined by $[\sigma, \tau] = \sigma \circ \tau - (-1)^{\deg\sigma \deg\tau} \tau \circ \sigma$. The boundary operator $\partial : \mathrm{Der}_* M(Y) \to \mathrm{Der}_{*-1} M(Y)$ is defined by $\partial(\sigma) = [d, \sigma]$. Let $B\,\mathrm{aut}\, Y$ be the Dold-Lashof classifying space [2] for fibrations with fiber Y and $\tilde{B}\,\mathrm{aut}\, Y$ the universal covering. The differential graded Lie algebra $L = (\mathrm{Der}\, M(Y), \partial)$ is a model for $\tilde{B}\,\mathrm{aut}\, Y$ (see [8, page 313]).

Any fibration with fiber Y over a simply connected space B is the pullback of the universal fibration by a classifying map $h : B \to \tilde{B}\,\mathrm{aut}\, Y$. Let $Y \to E \to B$ be a fibration whose model [3, Section 15] is

$$M(B) = (\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, \overline{D}) = M(Y). \tag{2.1}$$

Take a basis a_i of $(\Lambda W)^+$, then there are derivations θ_i of ΛV such that for each $z \in V$, we have $D(z) = \overline{D}(z) + \sum_i a_i \theta_i(z)$. A differential graded algebra model for $\tilde{B}\,\mathrm{aut}\, Y$ is given by the cochain algebra $C^*(L)$ [3, 23(a)] on L, and a model for the classifying map of the fibration h is given by

$$h^* : C^*(L) = \mathrm{Hom}\left(\mathrm{Der}_{*-1} M(Y), \mathbb{Q}\right) \longrightarrow \Lambda W, \qquad h^*(\psi) = \sum_i a_i \psi(\theta_i) \tag{2.2}$$

(see [6, Section 9]). Put the derivation which sends a generator p to an element q and other generators to zero as (p, q) and the dual element with the degree shifted by $+1$ as $s(p, q)^*$.

Lemma 2.1. *The fibration $\mathcal{L}_X$ is the pullback of the universal fibration by a classifying map $h : X \to \tilde{B}\,\mathrm{aut}\,\Omega X$, where the model is given by $h^*(s(\overline{v}_i, \overline{v}_j)^*) = \pm_{i,j} \partial dv_i/\partial v_j$ for $v_i, v_j \in V$ and $h^*(\text{other}) = 0$.*

3. Proof. The category, $\mathrm{cat}(f)$, of a map $f : X \to Y$ is the least integer n such that X can be covered by $n+1$ open subsets U_i, for which the restriction of f to each U_i is null-homotopic. Note that $\mathrm{cat}(f) \le \mathrm{cat}(X)$. Recall that if $\eta : F \to E \to B$ is a simply connected fibration, then $\mathrm{genus}(\eta) = \mathrm{cat}(h)$ for the classifying map of η, $h : B \to \tilde{B}\,\mathrm{aut}\, F$ [5].

Proof of Theorem 1.1. Let $M(X) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n), d)$ with $\deg x_i$ even, $\deg y_i$ odd, $d(x_i) = 0$, and $d(y_i) = f_i \ne 0 \in \Lambda(x_1, \dots, x_n)$ for $i = 1, \dots, n$. Then $M(\Omega X) = (\Lambda(\overline{x}_1, \dots, \overline{x}_n, \overline{y}_1, \dots, \overline{y}_n), 0)$ with $\deg\overline{v} = \deg v - 1$ for any element v. The minimal model of the space LX of free loops on X is given by

$$M(LX) = (\Lambda(x_1, \dots, x_n, y_1, \dots, y_n, \overline{x}_1, \dots, \overline{x}_n, \overline{y}_1, \dots, \overline{y}_n), \delta), \tag{3.1}$$

where $\delta x_i = \delta\overline{x}_i = 0$, $\delta y_i = dy_i = f_i$, $\delta\overline{y}_i = -\sum_{j=1}^n \partial f_i/\partial x_j \cdot \overline{x}_j$. Then we see from Lemma 2.1 that

$$h^*(s(\overline{y}_i, \overline{x}_j)^*) = -\frac{\partial f_i}{\partial x_j} \quad \text{for } 1 \le i,\ j \le n,\ h^*(\text{other}) = 0. \tag{3.2}$$

Let J be the determinant of the matrix whose (i,j)-component is $s(\overline{y}_i,\overline{x}_j)^*$. Then $(-1)^n h^*(J)$ is the Jacobian $|(\partial f_i/\partial x_j)_{1\le i,\, j\le n}|$ of $f_1,\dots,f_n$ and it is a cocycle which is not cohomologous to zero in $M(X)$ [7, Theorem B]. Therefore, as in [4, page 598(2)],

$$\operatorname{genus}(\mathcal{L}_X) = \operatorname{cat}(h) \ge \operatorname{nil}(\operatorname{Im}\tilde{H}(h^*)) \ge n, \tag{3.3}$$

where nil$R$ is the least integer n such that $R^{n+1} = 0$ for a ring R and $\tilde{H}(h^*)$ is the induced morphism in reduced cohomology. □

Corollary 3.1. *If X is an F_0-space of n variables with* $\operatorname{cat}(X) = n$, *then* $\operatorname{genus}(\mathcal{L}_X) = n$.

Example 3.2. Let $X = S^{2n} \vee S^{2n} \cup e^{4n} \not\simeq_0 S^{2n} \vee S^{2n} \vee S^{4n}$. X is an F_0-space where $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,x_2]/({x_1}^2 + a{x_2}^2, x_1x_2)$ with some $a \ne 0 \in \mathbb{Q}$ and $\deg x_i = 2n$. Then from Theorem 1.1 and [3, Lemma 27.3], $2 \le \operatorname{genus}(\mathcal{L}_X) \le \operatorname{cat}(X) \le \operatorname{cat}(S^{2n} \vee S^{2n}) + 1 = 2$, that is, $\operatorname{genus}(\mathcal{L}_X) = \operatorname{cat}(X) = 2$.

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Toshihiro Yamaguchi: Faculty of Education, Kochi University, Kochi 780-8520, Japan
E-mail address: tyamag@cc.kochi-u.ac.jp