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ON THE CLASS OF QS-ALGEBRAS

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We consider some fundamental properties of QS-algebras and show that (1) the theory of QS-algebras is logically equivalent to the theory of *Abelian groups*, that is, each theorem of QS-algebras is provable in the theory of Abelian groups, and conversely, each theorem of Abelian groups is provable in the theory of QS-algebras; and (2) a G-part $G(X)$ of a QS-algebra X is a normal subgroup generated by the class of all elements of order 2 of X when it is considered as a group.

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1. Introduction. In [3], the notion of Q-algebras is introduced and some fundamental properties are established. The algebras are extensions of the BCK/BCI-algebras which were proposed by Y. Imai and K. Iséki in 1966. It is usually important to generalize the algebraic structures. Neggers and Kim [4] introduced a class of algebras which is related to several classes of algebras such as BCK/BCI/BCH-algebras. They call them *B-algebras*, and they proved that every group $(X;\circ,0)$ determines a B-algebra $(X;*,0)$, which is called the *group-derived B-algebra*. Conversely, in [2], we prove that every B-algebra is group-derived and hence that the class of B-algebras and the class of all groups are the same. Ahn and Kim [1] proposed the notion of QS-algebras which is also a generalization of BCK/BCI-algebras and obtained several results. Here, we consider some fundamental properties of QS-algebras and show that

(1) the theory of QS-algebras is logically equivalent to the theory of *Abelian groups*, that is, each theorem of QS-algebras is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is provable in the theory of QS-algebras;

(2) a subset $G(X)$ called G-part of a QS-algebra X is a normal subgroup which is generated by the class of all elements of order 2.

2. Preliminaries. A *QS-algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(QS1) $x * x = 0$,

(QS2) $x * 0 = x$,

(QS3) $(x * y) * z = (x * z) * y$,

(QS4) $(x * y) * (x * z) = y * z$,

for all x, y, z in X.

Example 2.1. (1) Let $X = \{0,1,2\}$ be a set with an operation $*$ defined as follows:

$$\begin{array}{c|ccc} * & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 1 & 0 \end{array} \tag{2.1}$$

Then $(X;*,0)$ is a QS-algebra.

(2) Let X be the set of all integers. Define a binary operation $*$ on X by

$$x * y := x - y. \tag{2.2}$$

Then $(X;*,0)$ is a QS-algebra.

We note that these examples are both Abelian groups and the operation $*$ corresponds to the minus operation "$-$". In the case of (1), X can be considered as the set $\mathbb{Z}_3$ of integers of modulo 3 and the operation $*$ as a minus "$-$" modulo operation. It seems that any Abelian group gives an example of a QS-algebra. In fact, we can prove the fact.

Theorem 2.2. *Let $(X;\cdot,-1,e)$ be an Abelian group. If $x * y = x \cdot y^{-1}$ is defined and $0 = e$, then $(X;*,0)$ is a QS-algebra.*

Proof. We only show that the conditions (QS3) and (QS4) of QS-algebras are satisfied. For the case of (QS3), since X is an Abelian group,

$$\begin{aligned} (x * y) * z &= (x \cdot y^{-1}) \cdot z^{-1} \\ &= x \cdot (y^{-1} \cdot z^{-1}) \\ &= x \cdot (z^{-1} \cdot y^{-1}) \\ &= (x \cdot z^{-1}) \cdot y^{-1} \\ &= (x * z) * y. \end{aligned} \tag{2.3}$$

For the case of (QS4), we also have

$$\begin{aligned} (x * y) * (x * z) &= (x \cdot y^{-1}) \cdot (x \cdot z^{-1})^{-1} \\ &= (x \cdot y^{-1}) \cdot (z \cdot x^{-1}) \\ &= x \cdot x^{-1} \cdot z \cdot y^{-1} \\ &= z \cdot y^{-1} \\ &= z * y. \end{aligned} \tag{2.4}$$

□

The theorem means that every Abelian group $(X;\cdot,-1,e)$ determines a QS-algebra $(X;*,0)$; in other words, any Abelian group can be considered as a QS-algebra. Conversely, we will show in the next section that any QS-algebra determines an Abelian group, that is, every QS-algebra can be considered as an Abelian group. Hence, we are able to conclude that in this sense, the class of QS-algebras coincides with the class of Abelian groups.

3. Abelian groups can be derived from QS-algebras. We show that every QS-algebra determines an Abelian group. In order to do so, it is sufficient to construct an Abelian group from any QS-algebra. We need some lemmas to prove that.

Let $(X;*,0)$ be a QS-algebra.

Lemma 3.1. *For all $x,y,z \in X$, if $x*y=z$, then $x*z=y$.*

Proof. Suppose that $x*y=z$. Then, since X is a QS-algebra, we have $x*z=(x*0)*(x*y)=y*0=y$. □

It follows from the above that the condition (QS4)′ is established in any QS-algebra:
(QS4)′ $(x*z)*(y*z)=x*y$.

Corollary 3.2. *If $x*y=0$, then $x=y$.*

Let $B(X)=\{x\in X \mid 0*x=0\}$. A QS-algebra X is called *p-semisimple* if $B(X)=\{0\}$ (cf. [1]). We can show that every QS-algebra is p-semisimple.

Corollary 3.3. *Every QS-algebra is p-semisimple.*

Proof. Suppose that X is a QS-algebra. For all elements $x\in X$, since

$$\begin{aligned} x\in B(X) &\Longleftrightarrow 0*x=0 \\ &\Longleftrightarrow x=0 \quad (\text{by Corollary 3.2}), \end{aligned} \tag{3.1}$$

we can conclude that X is p-semisimple. □

Remark 3.4. It is proved in [1] that every *associative* QS-algebra is p-semisimple. The corollary above means that the assumption of associativity is superfluous.

Lemma 3.5. $0*(x*y)=y*x$.

Proof. $0*(x*y)=(x*x)*(x*y)=y*x$. □

Corollary 3.6. $0*(0*x)=x$.

Lemma 3.7. $x*(0*y)=y*(0*x)$.

Proof. Since

$$\begin{aligned} 0*(x*(y*(0*x))) &= (y*(0*x))*x \quad (\text{by Lemma 3.5}) \\ &= (y*x)*(0*x) \quad (\text{by (QS3)}) \\ &= y*0 \quad (\text{by (QS4)}') \\ &= y, \end{aligned} \tag{3.2}$$

we have $0*(0*(x*(y*(0*x))))=0*y$. It follows from Corollary 3.6 that $x*(y*(0*x))=0*y$ and hence $x*(0*y)=y*(0*x)$ by Lemma 3.1. □

These lemmas provide a proof that we can construct an Abelian group $(X;\cdot,e)$ from a QS-algebra $(X;*,0)$.

Theorem 3.8. *Let $(X;*,0)$ be a QS-algebra. If $x\cdot y=x*(0*y)$ is defined, $x^{-1}=0*x$, and $e=0$, then the structure $(X;\cdot,{}^{-1},e)$ is an Abelian group.*

Proof. We only show that the structure $(X;\cdot,-1,e)$ satisfies the conditions of associativity and of commutativity with respect to the operation "$\cdot$".

For associativity, we have

$$\begin{aligned}(x\cdot y)\cdot z &= \big(x*(0*y)\big)*(0*z)\\ &= \big(y*(0*x)\big)*(0*z) \quad \text{(by Lemma 3.7)}\\ &= \big(y*(0*z)\big)*(0*x)\\ &= x*\big(0*\big(y*(0*z)\big)\big) \quad \text{(by Lemma 3.7)}\\ &= x\cdot(y\cdot z).\end{aligned} \tag{3.3}$$

For commutativity, we also have $x\cdot y = x*(0*y) = y*(0*x) = y\cdot x$. □

Combining Theorems 2.2 and 3.8, we can conclude that the class of QS-algebras coincides with the class of Abelian groups.

In the following, we will describe our results in greater detail. We can show that each theorem of QS-algebras is translated to a formula of $\mathcal{AG}$ which is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is proved in the theory of QS-algebras. To present our theorem precisely, we will develop the formal theories of QS-algebras and Abelian groups. Let $\mathcal{QS}$ and $\mathcal{AG}$ be the theories of QS-algebras and Abelian groups, respectively. Theories consist of languages and axioms. At first, we define languages of these theories which are needed to present statements formally in their theories. By $\mathcal{L}(\mathcal{QS})$ (or $\mathcal{L}(\mathcal{AG})$), we mean a language of the theory $\mathcal{QS}$ of QS-algebras (or the theory $\mathcal{AG}$ of groups). We define them as follows.

The language of the theory of $\mathcal{QS}$ of QS-algebras consists of

(lq1) countable variables $x,y,z,\ldots$,

(lq2) binary operation symbol $*$,

(lq3) constant symbol 0;

and the language of the theory of $\mathcal{AG}$ of QS-algebras consists of

(lg1) countable variables $x,y,z,\ldots$,

(lg2) binary operation symbol $\circ$,

(lg3) unary operation symbol -1,

(lg4) constant symbol e.

Next we define *terms* which represent objects in the theory. By $\mathcal{T}(\mathcal{QS})$ (or $\mathcal{T}(\mathcal{AG})$) we mean the set of terms of $\mathcal{QS}$ (or $\mathcal{AG}$). Terms are defined as follows.

For terms of $\mathcal{QS}$,

(tb1) each variable is a term,

(tb2) the constant 0 is a term,

(tb3) if u and v are terms, then $u*v$ is a term.

For terms of $\mathcal{AG}$,

(tg1) each variable is a term,

(tg2) the constant e is a term,

(tg3) if u and v are terms, then so are $u\circ v$ and u^{-1}.

We also define *formulas* which represent statements in each theory. Formulas of $\mathcal{QS}$ (or $\mathcal{AG}$) are defined as the forms of $s=t$, where $s,t\in\mathcal{T}(\mathcal{QS})$ (or $s,t\in\mathcal{T}(\mathcal{AG})$). By $\mathcal{F}(\mathcal{QS})$

(or $\mathcal{F}(\mathcal{AG})$) we mean the set of formulas of $\mathcal{QS}$ (or $\mathcal{AG}$). We denote formulas simply by $A, B, C, \ldots$.

As to the axioms of QS-algebras, we list the following:

(QS1) $x * x = 0$,

(QS2) $x * 0 = x$,

(QS3) $(x * y) * z = (x * z) * y$,

(QS4) $(x * y) * (x * z) = y * z$.

For the axioms of Abelian groups, we use the following:

(G1) $x \circ (y \circ z) = (x \circ y) \circ z$,

(G2) $x \circ e = e \circ x = x$,

(G3) $x \circ x^{-1} = x^{-1} \circ x = e$,

(G4) $x \circ y = y \circ x$.

Two formal theories $\mathcal{QS}$ and $\mathcal{AG}$ have the same rules of inference concerning "equality," for they have no predicate symbols.

Rules of inference. For all terms $s, t, w, s_1, s_2, \ldots \in \mathcal{T}(\mathcal{QS})$ (or $\mathcal{T}(\mathcal{AG})$),

$$s = s, \qquad \frac{s = t}{t = s}, \qquad \frac{s = t,\ t = w}{s = t}, \qquad \frac{s_1 = t_1, \ldots, s_n = t_n}{\phi(s_1, \ldots, s_n) = \phi(t_1, \ldots, t_n)}, \tag{3.4}$$

where $\phi(x_1, \ldots, x_n)$ is a term of $\mathcal{T}(\mathcal{QS})$ (or $\mathcal{T}(\mathcal{AG})$) whose variables are contained in $\{x_1, \ldots, x_n\}$.

We are now ready to present a formal theory of Abelian groups and QS-algebras. Let Γ be a subset of formulas of $\mathcal{F}(\mathcal{QS})$ (or $\mathcal{F}(\mathcal{AG})$). By

$$\Gamma \vdash_{\mathcal{QS}} A (\Gamma \vdash_{\mathcal{AG}} A), \tag{3.5}$$

we mean that there is a finite sequence of formulas $A_1, A_2, \ldots, A_n$ of $\mathcal{F}(\mathcal{QS})(\mathcal{F}(\mathcal{AG}))$ such that for each i,

(1) A_i is an axiom of $\mathcal{QS}(\mathcal{AG})$,

(2) $A_i \in \Gamma$,

(3) there exists $j_i, \ldots, j_k$ $(j_1, \ldots, j_k < i)$ such that

$$\frac{A_{j_1}, \ldots, A_{j_k}}{A}. \tag{3.6}$$

We say that A is provable from Γ in $\mathcal{QS}(\mathcal{AG})$ when $\Gamma \vdash_{\mathcal{AG}} A (\Gamma \vdash_{\mathcal{AG}} A)$. In particular, in case of $\Gamma = \varnothing$, we say that A is a theorem of $\mathcal{QS}(\mathcal{AG})$ and simply denote it by $\vdash_{\mathcal{QS}} A (\vdash_{\mathcal{AG}} A)$.

As an example, we present the following which is called a cancelation rule in the theory of groups:

$$x \circ y = z \circ x \vdash_{\mathcal{AG}} y = z. \tag{3.7}$$

Indeed, we have the following finite sequence of formulas:

$$\begin{gathered} x\circ y=z\circ x,\quad z\circ x=x\circ z,\quad x\circ y=x\circ z,\quad x^{-1}\circ(x\circ y)=x^{-1}\circ(x\circ z),\\ x^{-1}\circ(x\circ y)=(x^{-1}\circ x)\circ y,\quad x^{-1}\circ(x\circ z)=(x^{-1}\circ x)\circ z,\\ (x^{-1}\circ x)\circ y=(x^{-1}\circ x)\circ z,\quad x^{-1}\circ x=e,\quad e\circ y=e\circ z,\\ e\circ y=y,\quad e\circ z=z,\quad y=z. \end{gathered} \tag{3.8}$$

Next, we define two maps ξ from the theory $\mathcal{QS}$ of QS-algebras to the theory $\mathcal{AG}$ of Abelian groups and η from $\mathcal{AG}$ to $\mathcal{QS}$ as follows. For $\mathcal{T}(\mathcal{QS})$,

$$\begin{gathered} \xi(x)\equiv x\quad \text{for each variable } x,\\ \xi(0)\equiv e,\\ \xi(s*t)\equiv\xi(s)\circ\xi(t^{-1}), \end{gathered} \tag{3.9}$$

and for $\mathcal{F}(\mathcal{QS})$,

$$\begin{gathered} \xi(s=t)\equiv\xi(s)=\xi(t),\\ \xi(s=t\Longrightarrow s'=t')\equiv\xi(s=t)\Longrightarrow\xi(s'=t'), \end{gathered} \tag{3.10}$$

where $s,s',t,t'\in\mathcal{T}(\mathcal{QS})$.

Conversely, we define a map $\eta:\mathcal{AG}\to\mathcal{QS}$ as follows. For $\mathcal{T}(\mathcal{AG})$,

$$\begin{gathered} \eta(x)\equiv x\quad \text{for each variable } x,\\ \eta(e)\equiv 0,\\ \eta(s^{-1})\equiv 0*(\eta(s)),\\ \eta(s\circ t)\equiv\eta(s)*(0*\eta(t)), \end{gathered} \tag{3.11}$$

and for $\mathcal{F}(\mathcal{AG})$,

$$\begin{gathered} \eta(s=t)\equiv\eta(s)=\eta(t),\\ \eta(s=t\Longrightarrow s'=t')\equiv\eta(s=t)\Longrightarrow\eta(s'=t'), \end{gathered} \tag{3.12}$$

where $s,s',t,t'\in\mathcal{T}(\mathcal{AG})$.

We are now ready to state our theorem. It is as follows.

Main theorem 3.9. (1) *For every* $\Gamma\cup\{A\}\in\mathcal{F}(\mathcal{QS})$, *if* $\Gamma\vdash_{\mathcal{QS}}A$, *then* $\xi(\Gamma)\vdash_{\mathcal{AG}}\xi(A)$; *conversely,*

(2) *for every* $\Gamma\cup\{A\}\in\mathcal{F}(\mathcal{AG})$, *if* $\Gamma\vdash_{\mathcal{AG}}A$, *then* $\eta(\Gamma)\vdash_{\mathcal{QS}}\eta(A)$; *moreover,*

(3) $\Gamma\vdash_{\mathcal{QS}}\eta\xi(A)$ *if and only if* $\Gamma\vdash_{\mathcal{QS}}A$ *for every* $\Gamma\cup\{A\}\in\mathcal{F}(\mathcal{QS})$;

(4) $\Gamma\vdash_{\mathcal{AG}}\xi\eta(A)$ *if and only if* $\Gamma\vdash_{\mathcal{AG}}A$ *for every* $\Gamma\cup\{A\}\in\mathcal{F}(\mathcal{AG})$.

This means that each theorem of QS-algebras can be translated immediately to that of groups and conversely, every theorem of groups is applied to that of QS-algebras.

At first we will establish the former part, that is, if $\Gamma\vdash_{\mathcal{QS}}A$, then $\xi(\Gamma)\vdash_{\mathcal{AG}}\xi(A)$.

Theorem 3.10. *For $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{QS})$, if $\Gamma \vdash_{\mathcal{QS}} A$, then $\xi(\Gamma) \vdash_{\mathcal{AG}} \xi(A)$.*

Proof. It is sufficient to show that for every axiom A of QS-algebras, $\xi(A)$ is provable in the theory $\mathcal{AG}$ of Abelian groups. For the sake of simplicity, we treat only the case of axiom (QS3): $(x*y)*z = x*(z*(0*y))$. Other cases can be proved similarly.

Since

$$\begin{aligned} \xi((x*y)*z) &= (x\circ y^{-1})\circ z^{-1}, \\ \xi(x*(z*(0*y))) &= x\circ\left\{z\circ(y^{-1})^{-1}\right\}^{-1}, \end{aligned} \tag{3.13}$$

we have to show that

$$(x\circ y^{-1})\circ z^{-1} = x\circ\left\{z\circ(y^{-1})^{-1}\right\}^{-1}. \tag{3.14}$$

We have the following:

$$\begin{aligned} x\circ\left\{z\circ(y^{-1})^{-1}\right\}^{-1} &= x\circ(z\circ y)^{-1} \\ &= x\circ(y^{-1}\circ z^{-1}) \\ &= (x\circ y^{-1})\circ z^{-1}. \end{aligned} \tag{3.15}$$

Hence, if $\Gamma \vdash_{\mathcal{QS}} A$, then $\xi(\Gamma) \vdash_{\mathcal{AG}} \xi(A)$. □

Conversely, we can show that if $\Gamma \vdash_{\mathcal{AG}} A$, then $\eta(\Gamma) \vdash_{\mathcal{QS}} \eta(A)$.

Theorem 3.11. *For $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{AG})$, if $\Gamma \vdash_{\mathcal{AG}} A$, then $\eta(\Gamma) \vdash_{\mathcal{QS}} \eta(A)$.*

Proof. As above, it is sufficient to show that $\eta(A)$ is provable in the theory $\mathcal{QS}$ of QS-algebras for every axiom A of the theory of Abelian groups.

For the case of (G1), we have to show that

$$(x*(0*y))*(0*z) = x*(0*(y*(0*z))) \tag{3.16}$$

because $\eta((x\circ y)\circ z) = (x*(0*y))*(0*z)$ and $\eta(x\circ(y\circ z)) = x*(0*(y*(0*z)))$. By the proposition above, we have

$$\begin{aligned} (x*(0*y))*(0*z) &= x*((0*z)*(0*(0*y))) \\ &= x*((0*z)*y) \\ &= x*(0*(y*(0*z))). \end{aligned} \tag{3.17}$$

Other cases are proved easily, so we omit their proofs.

The theorem can be proved completely. □

Moreover, it follows from Lemma 3.5 and Corollary 3.6 that we have $\vdash_{\mathcal{QS}} \eta\xi(t) = t$ for every term $t \in \mathcal{T}(\mathcal{QS})$. For the case of Abelian groups, it is easy to prove that $\vdash_{\mathcal{AG}} \xi\eta(s) = s$ for every term $s \in \mathcal{T}(\mathcal{AG})$. Thus we have the following.

Theorem 3.12. *For these maps,*

(1) $\Gamma \vdash_{\mathcal{QS}} \eta\xi(A)$ *if and only if* $\Gamma \vdash_{\mathcal{QS}} A$ *for all* $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{QS})$,

(2) $\Gamma \vdash_{\mathcal{AG}} \xi\eta(A)$ *if and only if* $\Gamma \vdash_{\mathcal{AG}} A$ *for all* $\Gamma \cup \{A\} \in \mathcal{F}(\mathcal{AG})$.

4. Some properties. In this section, we prove other properties of QS-algebras, especially properties about the G-part and *mediality*. That is,

(1) the G-part $G(X)$ of a QS-algebra X is a normal subgroup generated by the class of all elements of order 2 of X;

(2) every QS-algebra X is *medial*, that is, it satisfies the condition

$$(x*y)*(z*u)=(x*z)*(y*u) \tag{4.1}$$

for all elements $x,y,z,u\in X$.

Let X be a QS-algebra. A subset $G(X)=\{x\in X \mid 0*x=x\}$ is called a G-part of X. For the G-part of X, we have the following results.

Proposition 4.1. *If $x,y\in G(X)$, then $x*y=y*x$.*

Proof. Suppose $x,y\in G(X)$. Since $0*x=x$ and $0*y=y$, we have

$$\begin{aligned} x*y &= (0*x)*(0*y) \\ &= (0*(0*y))*x \quad \text{(by (QS3))} \\ &= y*x. \end{aligned} \tag{4.2}$$

□

Proposition 4.2. *If $x,y\in G(X)$, then $x*y\in G(X)$.*

Proof. Suppose that $x,y\in G(X)$. It follows that $(0*(x*y))*(x*y)=(y*x)*(x*y)=(x*y)*(x*y)=0$ by Lemma 3.5. Thus we have $0*(x*y)=x*y$, that is, $x*y\in G(X)$. □

Since any QS-algebra X may be considered as an Abelian group, Proposition 4.2 implies that $G(X)$ is a (normal) subgroup of X. Moreover, since $x^2=x\cdot x=x*(0*x)=x*x=0$ for $x\in G(X)$, every nonunit element in $G(X)$ is of order 2. Hence, we can conclude that the G-part $G(X)$ is the normal subgroup generated by the class of all elements of order 2. It is easy to show that $G(X)=\{x\in X \mid x \text{ is of order } 2\}\cup\{0\}$.

For the statement (2) above, in [1, Theorem 3.6], it is proved that a QS-algebra X is medial if and only if the condition $x*(y*z)=(x*y)*(0*z)$ holds for all x, y, z $\in X$. On the other hand, by Lemma 3.7, we have $(x*y)*(0*z)=z*(0*(x*y))=z*(y*x)$. Thus X is medial if and only if the condition $x*(y*z)=z*(y*x)$ holds for all $x,y,z\in X$. By using this characterization of mediality, we will prove the following.

Theorem 4.3. *Every QS-algebra is medial.*

Proof. It is sufficient to show that $x*(y*z)=z*(y*x)$ holds for all $x,y,z\in X$. Since

$$\begin{aligned} x*(y*z) &= 0*(0*(x*(y*z))) \quad \text{(by Corollary 3.6)} \\ &= 0*((y*z)*x) \quad \text{(by Lemma 3.5)} \\ &= 0*((y*x)*z) \quad \text{(by (QS3))} \\ &= z*(y*x) \quad \text{(by Lemma 3.5)}, \end{aligned} \tag{4.3}$$

it follows that X is medial. □

5. Application. Let $V = \{x, y, z, \dots\}$ be a set of variables and $\mathbf{0}$ a constant. We define a *term* and *equation* as follows:

(1) $\mathbf{0}$ is a term;
(2) each variable in V is a term;
(3) if s, t are terms, then $s * t$ is also a term;
(4) if s, t are terms, then $s = t$ is an equation.

Thus, for example, $\mathbf{0}$, $\mathbf{0} * x$, $x * (\mathbf{0} * y)$, $x * y$ are terms and thus $\mathbf{0} = \mathbf{0} * x$, $x * (\mathbf{0} * y) = x * y$ are equations. By $t(x, y, \dots)$ we mean a term whose variables are in $\{x, y, \dots\}$. We say that an equation $s(x, y, \dots) = t(x, y, \dots)$ is satisfied in a QS-algebra X when for all elements $a, b, \dots \in X$, we have $u^X(a, b, \dots) = v^X(a, b, \dots)$. In particular, an equation $t(x, y, \dots) = \mathbf{0}$ is said to be satisfied in X if $t^X(a, b, \dots) = 0$ for all elements $a, b, \dots \in X$. In the following, by $t(a, b, \dots)$ we mean an element $t^X(a, b, \dots)$ which is an interpretation of a term $t(x, y, \dots)$ in X, that is, $t(a, b, \dots)$ is an abbreviation of $t^X(a, b, \dots)$.

We also define a condition (C) which plays an important role to develop our theory:

(C) for all x and for all y, there exists $t(x, y)$ such that

$$(\mathbf{0} * (\mathbf{0} * x)) * (\mathbf{0} * (\mathbf{0} * y)) = \mathbf{0} * (\mathbf{0} * t(x, y)), \quad t(x, x) = \mathbf{0}. \tag{5.1}$$

By using condition (C), we have the following theorem which shows the relation between Q-algebras and QS-algebras.

Theorem 5.1. *Let $(X; *, 0)$ be a Q-algebra. $(X; *, 0)$ satisfies condition (C) if and only if $(X^*; *, 0)$ is a QS-algebra, where $X^* = \{0 * (0 * a) \mid a \in X\}$.*

Proof. *If part.* For all $u, v \in X^*$, there exist $a, b \in X$ such that $u = 0 * (0 * a)$, $v = 0 * (0 * b)$. It follows from condition (C) that $u * v \in X^*$ and that X^* is a subalgebra of X. Hence, X^* is a Q-algebra. We define $u \cdot v = u * (0 * v)$ and $u^{-1} = 0 * u$. Since $u, v, 0 \in X^*$, we have $u \cdot v \in X^*$. Moreover for this operation, we can show that

(i) $u \cdot v = v \cdot u$,
(ii) $u \cdot 0 = 0 \cdot u = u$,
(iii) $u \cdot (0 * u) = (0 * u) \cdot u = 0$,
(iv) $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.

For the sake of simplicity, we only prove the case of (iv). Before doing so, we note the following result: $(u \cdot v) \cdot w = (u \cdot w) \cdot v$ for all $u, v, w \in X^*$. Because

$$\begin{aligned}(u \cdot v) \cdot w &= (u \cdot v) * (0 * w) \\ &= (u * (0 * v)) * (0 * w) \\ &= (u * (0 * w)) * (0 * v) \\ &= (u \cdot w) \cdot v,\end{aligned} \tag{5.2}$$

it follows from the result that $(u \cdot v) \cdot w = (v \cdot u) \cdot w = (v \cdot w) \cdot u = u \cdot (v \cdot w)$.

Thus the above means that $(X^*; \cdot, -1, 0)$ is an Abelian group. For this group, if we define $u \circ v = u \cdot (0 * v)$, then $(X^*; \circ, 0)$ is a QS-algebra. Clearly we have $u \circ v = u \cdot (0 * v) = u * (0 * (0 * v)) = u * v$ for all $u, v \in X^*$. That is, $(X^*; *, 0)$ is a QS-algebra.

Only if part. Conversely, we suppose that $(X^*;*,0)$ is a QS-algebra. For all $u,v \in X^*$, there exist $a,b \in X$ such that $u = 0*(0*a)$, $v = 0*(0*b)$. Since $u*v \in X^*$, $u*v$ has to have a form of $0*(0*t(a,b))$. This means that X^* satisfies the condition

$$\forall x \forall y \exists t(x,y) \quad (0*(0*x))*(0*(0*y)) = 0*(0*t(x,y)). \tag{5.3}$$

It is obvious that $t(a,a) = 0$ for all $a \in X$, that is, $t(x,x) = 0$. Thus X^* satisfies condition (C). □

We consider some cases of $t(x,y)$ as corollaries to the theorem. First of all, let $t(x,y)$ be a form of $x*y$, that is, $t(x,y) = x*y$. In this case, condition (C) has a form of

$$(0*(0*x))*(0*(0*y)) = 0*(0*(x*y)). \tag{5.4}$$

For a map $f : X \to X^*$ defined by $f(x) = 0*(0*x)$, since $(x,y) \in \operatorname{Ker} f$ if and only if $0*(0*x) = 0*(0*y)$, we have that $X/\operatorname{Ker} f$, the quotient Q-algebra modulo $\operatorname{Ker} f$, is isomorphic to X^*, that is, $X/\operatorname{Ker} f \cong X^*$. Hence, we have the following.

Corollary 5.2. *If $f : X \to X^*$ is a map defined by $f(x) = 0*(0*x)$, then $X/\operatorname{Ker} f \cong X^*$.*

We define a term $t^n(x,y)$ for all nonnegative integers n as follows:

$$\begin{gathered} t^0(x,y) = 0*(x*y), \\ t^n(x,y) = t^{n-1}(x,y)*(0*(0*(x*y))). \end{gathered} \tag{5.5}$$

In this case, the corresponding condition (C_n) is

(C_n)

$$(0*(0*x))*(0*(0*y)) = 0*(0*t^n(x,y)). \tag{5.6}$$

We now have the following result as to condition (C_n).

Corollary 5.3. *Let X be a QS-algebra. If X satisfies condition (C_n), then X^* is an Abelian group in which every element has order at most $(n+2)$.*

Proof. For condition (C_n), if we take $y = 0$, then we have $0*(0*x) = 0*(0*t^n(x,0))$, that is,

$$0*(0*x) = 0*[0*\{((0*x)*(0*(0*x)))*(0*(0*x))*\cdots*(0*(0*x))\}]. \tag{5.7}$$

Since any element $u \in X^*$ has a form of $0*(0*a)$ for some element $a \in X$, it follows from (C_n) that

$$\begin{aligned} u &= 0*[0*\{((0*a)*(0*(0*a)))*(0*(0*a))*\cdots*(0*(0*a))\}] \\ &= 0*[0*\{((\cdots((0*a)*u)*u)*\cdots)*u\}] \\ &= ((\cdots((0*a)*u)*u)*\cdots)*u \\ &= ((\cdots((0*u)*u)*u)*\cdots)*u. \end{aligned} \tag{5.8}$$

On the other hand, since $(0*u)*u = (0*u)*(0*(0*u)) = u^{-1}\cdot u^{-1}$ in the Abelian group X^*, we have

$$u = (u^{-1}\cdot u^{-1})\cdot u^{-1}\cdots u^{-1} = (u^{-1})^{n+1} = u^{-(n+1)} \tag{5.9}$$

and hence $u^{n+2} = 0$. This means that each element of X^* has order at most $n+2$. □

As the last case, we suppose $t(x,y) = (x*y)*(y*x)$. Since condition (C) is

$$(0*(0*x))*(0*(0*y)) = 0*(0*((x*y)*(y*x))) \tag{5.10}$$

in this case, if we take $y = 0$, then we have the condition

$$0*(0*x) = 0*(0*(x*(0*x))). \tag{5.11}$$

This implies that

$$0*(0*a) = 0*(0*(a*(0*a))) \tag{5.12}$$

for all $a \in X$. In particular, any element $u \in X^*$ satisfies the condition. Hence, since $u = 0*(0*u)$ for all elements $u \in X^*$, we have in the Abelian group X^*

$$u = u*(0*u) = u\cdot u. \tag{5.13}$$

This means that every element of X^* is an idempotent. But $u\cdot uu$ means $u = 0$ and $X^* = \{0\}$ is trivial.

Corollary 5.4. *Let X be a Q-algebra. If X satisfies the condition*

$$(0*(0*x))*(0*(0*y)) = 0*(0*((x*y)*(y*x))), \tag{5.14}$$

then X^* is an Abelian group in which every element is an idempotent.

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