AN EXTENSION OF $q$-ZETA FUNCTION

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We will define the extension of $q$-Hurwitz zeta function due to Kim and Rim (2000) and study its properties. Finally, we lead to a useful new integral representation for the $q$-zeta function.

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1. Introduction. Let $0 < q < 1$ and for any positive integer $k$, define its $q$-analogue $[k]_q = (1 - q^k)/(1 - q)$. Let $\mathbb{C}$ be the field of complex numbers. The $q$-zeta function due to T. Kim was defined as

$$
\zeta_{q}(h)(s) = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_q^s} + (q - 1) \frac{1}{1-s} \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_q^{s-1}}
$$

(1.1)

for any $s, h \in \mathbb{C}$ (cf. [3, 4]). This function can be considered on the spectral zeta function of the quantum group $SU_q(2)$ (cf. [2, 4]). Also, the $q$-zeta function $\zeta_{q}(h)(s)$ was studied at negative integers (see [4]). In this note, we lead to a useful new integral representation for the $q$-zeta function $\zeta_{q}(h)(s)$. Finally, we define the extension of $q$-Hurwitz zeta function, and study its properties.

2. $q$-zeta functions. For $q \in \mathbb{C}$ with $|q| < 1$, we define $q$-Bernoulli polynomials as follows:

$$
F_{q}(h)(t, x) = \sum_{n=0}^{\infty} \frac{\beta_{n,q}(x)}{n!} t^n
$$

$$
e^{(1/(1-q))t} \sum_{j=0}^{\infty} \frac{j+h}{[j+h]_q} (-1)^j q^j x \left( \frac{1}{1-q} \right)^j \frac{t^j}{j!}
$$

$$
= -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_q t} + (1-q) h \sum_{l=0}^{\infty} q^{lh} e^{[l+x]_q t}
$$

(2.1)

for $h \in \mathbb{Z}, x \in \mathbb{C}$ (cf. [2, 4]). In the case $x = 0$, $\beta_{n,q}(x) = \beta_{n,q}(0)$ will be called the $q$-Bernoulli numbers (cf. [4]). By (2.1), we easily see that...
\[ \beta_{n,q}^{(h)}(x) = \sum_{j=0}^{m} \binom{m}{j} [x]_{q}^{n-j} q^{jx} \beta_{j,q}^{(h)} \]

\[ = \left( \frac{1}{1-q} \right)^{n} \sum_{j=0}^{n} \binom{n}{j} (-1)^{j} j + h \left[ \frac{j + h}{q} \right] q^{jx} \quad \text{(cf. [2])}, \]

where \( \binom{n}{j} \) is a binomial coefficient.

Thus we note that

\[ q^{h}(q^{\beta(h)} + 1)^{n} - \beta_{n,q}^{(h)} = \delta_{1,n}, \]

where we use the usual convention about replacing \( (\beta(h))^{n} \) by \( \beta_{n,q}^{(h)} \) and \( \delta_{1,n} \) is the Kronecker symbol.

**Example 2.1.**

\[ \beta_{0}^{(2)} = \frac{2}{[2]}, \quad \beta_{1}^{(2)} = -2q + \frac{1}{[2][3]}, \quad \beta_{2}^{(2)} = \frac{2q^{2}}{[3][4]}, \quad \beta_{3}^{(2)} = -\frac{q^{2}(q-1)(2[3]_{q} + q)}{[3][4][5]}, \quad \ldots \]

Let \( F_{q}^{(h)}(t) = \sum_{n=0}^{\infty} (\beta_{n,q}^{(h)}/n!)t^{n} \). Then we easily see that

\[ F_{q}^{(h)}(x,t) = e^{[x]_{q} t} F_{q}^{(h)}(q^{x} t) \]

\[ = -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_{q} t} + (1-q)h \sum_{l=0}^{\infty} q^{lh} e^{[l+x]_{q} t}. \]

By (2.1) and (2.5), we note that

\[ e^{-t} F_{q}^{(h)}(-qt) = qt \sum_{l=0}^{\infty} q^{l(h+1)} e^{-[l+1]_{q} t} + (1-q)h \sum_{l=0}^{\infty} q^{lh} e^{-[l+1]_{q} t}. \]

Thus we have

\[ \frac{1}{\Gamma(s)} \int_{0}^{\infty} q^{h} t^{s-2} e^{-t} F_{q}^{(h)}(-qt) \, dt = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s}} + (q-1) \frac{h+1-s}{1-s} \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s-1}}. \]

For \( h, s \in \mathbb{C} \), we define the \( q \)-zeta function as follows:

\[ \zeta_{q}^{(h)}(s) = \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s}} + (q-1) \frac{1-s+h}{1-s} \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_{q}^{s-1}} \quad \text{(cf. [1, 4])}. \]

Note that \( \zeta_{q}^{(h)}(s) \) is a meromorphic function for \( \text{Re}(s) > 1 \).

Let \( \Gamma(s) \) be the gamma function and let \( \mathbb{Z} \) be the set of integers. By (2.3), (2.7), and (2.8), we obtain the following.

For \( h, n(>1) \in \mathbb{Z} \), we have

\[ \zeta_{q}^{(h)}(1-n) = -\frac{q^{h}(q^{\beta(h)} + 1)^{n}}{n} = -\frac{\beta_{n,q}^{(h)}}{n}. \]
Let $x$ be any nonzero positive real number. Then we define the $q$-analogue of Hurwitz zeta function as follows:

$$
\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} q^{nh} \frac{n^s}{[n+x]_q^s} + \frac{h+1-s}{1-s} \sum_{n=0}^{\infty} q^{nh} \frac{1}{[n+x]_q^{s-1}}
$$

(2.10)

for $s,h \in \mathbb{C}$. By (2.5) and (2.10), we easily see that

$$
\zeta_q^{(h)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F_q^{(h)}(x,-t) \, dt.
$$

(2.11)

Thus we obtain the following: for $n \in \mathbb{N}, h \in \mathbb{Z}$, we have

$$
\zeta_q^{(h)}(1-n) = -\frac{\beta_q^{(h)}(x)}{n}
$$

(2.12)

because

$$
\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \beta_n^{(h)}(x) \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s+n-2} \, dt.
$$

(2.13)

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REFERENCES


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