ON DIFFERENTIAL SUBORDINATIONS FOR A CLASS 
OF ANALYTIC FUNCTIONS DEFINED 
BY A LINEAR OPERATOR

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We obtain several results concerning the differential subordination between analytic functions and a linear operator defined for a certain family of analytic functions which are introduced here by means of these linear operators. Also, some special cases are considered.

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1. Introduction. Let $\mathcal{A}_0$ be the class of normalized analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ which are defined in the unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ be the class of all analytic functions $p(z)$ with $p(0) = 1$ which are defined on $\Delta$. The class $\mathcal{P}$ of Carathéodory functions consists of functions $p(z) \in \mathcal{A}$ having positive real part. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (1.1)$$

their Hadamard product (or convolution) is defined, as usual, by

$$(f \ast g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g \ast f)(z). \quad (1.2)$$

Define the function $\phi(a,c;z)$ by

$$\phi(a,c;z) := \sum_{n=2}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0, -1, -2, \ldots ; z \in \Delta), \quad (1.3)$$

where $(x)_n$ is the Pochhammer symbol or the shifted factorial defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2) \cdots (x+n-1), & n \in \mathbb{N} := \{1,2,3,\ldots\}. \end{cases} \quad (1.4)$$

Corresponding to the function $\phi(a,c;z)$, Carlson and Shaffer [1] introduced a linear operator $L(a,c)$ on $\mathcal{A}_0$ by the following convolution:

$$L(a,c)f(z) := \phi(a,c;z) \ast f(z), \quad (1.5)$$
or, equivalently, by
\[
L(a, c) f(z) := z + \sum_{n=1}^{\infty} \left( \frac{a}{c} \right)^n a_{n+1} z^{n+1} \quad (z \in \Delta).
\tag{1.6}
\]

It follows from (1.6) that
\[
z (L(a, c) f(z))' = a L(a+1, c) f(z) - (a - 1) L(a, c) f(z).
\tag{1.7}
\]

For two functions \( f \) and \( g \) analytic in \( \Delta \), we say that the function \( f(z) \) is subordinate to \( g(z) \) in \( \Delta \), and write
\[
f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta),
\tag{1.8}
\]
if there exists a Schwarz function \( w(z) \), analytic in \( \Delta \) with
\[
w(0) = 0, \quad |w(z)| < 1 \quad (z \in \Delta),
\tag{1.9}
\]
such that
\[
f(z) = g(w(z)) \quad (z \in \Delta).
\tag{1.10}
\]

In particular, if the function \( g \) is univalent in \( \Delta \), the above subordination is equivalent to
\[
f(0) = g(0), \quad f(\Delta) \subset g(\Delta).
\tag{1.11}
\]

Over the past few decades, several authors have obtained criteria for univalence and starlikeness depending on bounds of the functionals \( z f'(z) / f(z) \) and \( 1 + z f''(z) / f'(z) \).

See [4, 5, 7] and the references in [7]. In [2, 6], certain results involving linear operators were considered. In this paper, we obtain sufficient conditions involving
\[
\frac{L(a+1, c) f(z)}{L(a, c) f(z)}, \quad \frac{L(a+2, c) f(z)}{L(a+1, c) f(z)}
\tag{1.12}
\]
for functions to satisfy the subordination
\[
\frac{L(a, c) f(z)}{L(a+1, c) f(z)} < q(z), \quad \left( \frac{L(a+1, c) f(z)}{L(a, c) f(z)} \right)^{\beta} < q(z) \quad (q(z) \in \mathcal{A}).
\tag{1.13}
\]

Also, we obtain sufficient conditions involving
\[
\frac{L(a+2, c) f(z)}{L(a+1, c) f(z)}, \quad \frac{L(a+1, c) f(z)}{z}
\tag{1.14}
\]
for functions to satisfy the subordination
\[
\left( \frac{L(a, c) f(z)}{z} \right)^{\beta} < q(z), \quad \frac{z}{L(a+1, c) f(z)} < q(z) \quad (q(z) \in \mathcal{A}).
\tag{1.15}
\]
Since \( L(n+1,1)f(z) = D^n f(z) \), where \( D^n f(z) \) is the Ruscheweyh derivative of \( f(z) \), our results can be specialized to the Ruscheweyh derivative and we omit these details. Note that the Ruscheweyh derivative of order \( \delta \) is defined by

\[
D_\delta f(z) := \frac{z}{(1-z)^{\delta+1}} \ast f(z) \quad (f \in \mathcal{A}_0; \ \delta > -1)
\]

or, equivalently, by

\[
D_\delta f(z) := z + \sum_{k=2}^{\infty} \left( \frac{\delta+k-1}{k+1} \right) a_k z^k \quad (f \in \mathcal{A}_0; \ \delta > -1).
\]

In our present investigation, we need the following result of Miller and Mocanu [3] to prove our main results.

**Theorem 1.1** (cf. [3, Theorem 3.4h, page 132]). Let \( q(z) \) be univalent in the unit disk \( \Delta \) and let \( \vartheta \) and \( \varphi \) be analytic in a domain \( \mathbb{D} \supset q(\Delta) \) with \( \varphi(w) \neq 0 \), when \( w \in q(\Delta) \). Set

\[
Q(z) := z q'(z) \varphi(q(z)), \quad h(z) := \vartheta(q(z)) + Q(z).
\]

Suppose that

1. \( Q \) is starlike univalent in \( \Delta \);
2. \( \Re(z h'(z)/Q(z)) > 0 \) for \( z \in \Delta \).

If \( p(z) \) is analytic in \( \Delta \), with \( p(0) = q(0) \), \( p(\Delta) \subset \mathbb{D} \), and

\[
\vartheta(p(z)) + z p'(z) \varphi(p(z)) < \vartheta(q(z)) + z q'(z) \varphi(q(z)),
\]

then \( p(z) < q(z) \) and \( q(z) \) is the best dominant.

2. Main results. We begin with the following.

**Theorem 2.1.** Let \( \alpha, \beta, \) and \( \gamma \) be real numbers, \( \beta \neq 0 \), and \( (1+a)\beta \gamma < 0 \). Let \( q(z) \in \mathcal{A} \) be univalent in \( \Delta \) and let it satisfy the following condition for \( z \in \Delta \):

\[
\Re \left( 1 + \frac{z q''(z)}{q'(z)} \right) \begin{cases} \frac{\beta + (1+a)\gamma}{\beta} & \text{if } \frac{\beta + \gamma(a+1)}{\beta} \geq 0, \\ 0 & \text{if } \frac{\beta + \gamma(a+1)}{\beta} \leq 0. \end{cases}
\]

(2.1)

If \( f(z) \in \mathcal{A}_0 \) and

\[
\frac{L(a,c)f(z)}{L(a+1,c)f(z)} \left\{ \alpha \frac{L(a+1,c)f(z)}{L(a,c)f(z)} + \beta \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \gamma \right\} < \frac{1}{a+1} \left\{ \alpha(a+1) + a \beta + [\beta + \gamma(a+1)]q(z) - \beta z q'(z) \right\},
\]

(2.2)
then
\[ \frac{L(a,c)f(z)}{L(a+1,c)f(z)} < q(z) \] (2.3)

and \( q(z) \) is the best dominant.

**Proof.** Define the function \( p(z) \) by
\[ p(z) := \frac{L(a,c)f(z)}{L(a+1,c)f(z)}. \] (2.4)

Then, clearly, \( p(z) \) is analytic in \( \Delta \). Also, by a simple computation, we find from (2.4) that
\[ \frac{zp'(z)}{p(z)} = \frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \frac{z(L(a+1,c)f(z))'}{L(a+1,c)f(z)}. \] (2.5)

By making use of the familiar identity (1.7) in (2.5), we get
\[ \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{1}{a+1} \left( 1 + \frac{a}{p(z)} - \frac{zp'(z)}{p(z)} \right). \] (2.6)

By using (2.4) and (2.6), we obtain
\[
\left[ \frac{\alpha L(a+1,c)f(z)}{L(a,c)f(z)} + \frac{\beta L(a+2,c)f(z)}{L(a+1,c)f(z)} + \gamma \right] \frac{L(a,c)f(z)}{L(a+1,c)f(z)} \\
= \left[ \frac{\alpha}{p(z)} + \frac{\beta}{a+1} \left( 1 + \frac{a}{p(z)} - \frac{zp'(z)}{p(z)} \right) + \gamma \right] p(z) \\
= \frac{1}{a+1} \left\{ (a+1)\alpha + a\beta + \left[ \beta + \gamma(a+1) \right] p(z) - \beta zp'(z) \right\}. \] (2.7)

In view of (2.7), the subordination (2.2) becomes
\[ [\beta + \gamma(a+1)]p(z) - \beta zp'(z) < [\beta + \gamma(a+1)]q(z) - \beta zq'(z) \] (2.8)

and this can be written as (1.19), where
\[ \vartheta(w) := [\beta + \gamma(a+1)]w, \quad \varphi(w) := -\beta. \] (2.9)

Note that \( \vartheta(w), \varphi(w) \) are analytic in \( \mathbb{C} \). Since \( \beta \neq 0 \), we have \( \varphi(w) \neq 0 \). Let the functions \( Q(z) \) and \( h(z) \) be defined by
\[
Q(z) := zq'(z)\varphi(q(z)) = -\beta zq'(z), \\
h(z) := \vartheta(q(z)) + Q(z) = [\beta + (a+1)\gamma]q(z) - \beta zq'(z). \] (2.10)

In light of hypothesis (2.1) stated in **Theorem 2.1**, we see that \( Q(z) \) is starlike and
\[
\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\gamma(a+1) + \beta}{-\beta} + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0. \] (2.11)

The result of **Theorem 2.1** now follows by an application of **Theorem 1.1**. \( \square \)
Note that
\[ L(1,1)f(z) = f(z), \]
\[ L(2,1)f(z) = zf'(z), \]
\[ L(3,1)f(z) = zf'(z) + \frac{z^2f''(z)}{2}. \] (2.12)

By taking \( a = c = 1 \) in Theorem 2.1 and after a change in the parameters, we have the following.

**Corollary 2.2.** Let \( \alpha \) be a real number, \( 1 + \alpha > 0 \), and let \( q(z) \) be univalent in \( \Delta \), and let it satisfy
\[ \Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} -\alpha & \text{if } \alpha \leq 0, \\ 0 & \text{if } \alpha > 0. \end{cases} \] (2.13)

If \( f \in A_0 \) and
\[ \frac{f(z)}{zf'(z)} \left\{ (1 - \alpha)\frac{zf'(z)}{f(z)} - \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \alpha \right\} < zq'(z) + \alpha q(z) - \alpha, \] (2.14)
then
\[ \frac{f(z)}{zf'(z)} < q(z) \] (2.15)
and \( q(z) \) is the best dominant.

If we take
\[ q(z) = 1 + \frac{\lambda}{1 + \alpha}z \] (2.16)
in Corollary 2.2, we obtain a recent result of Singh [7, Theorem 1(i), page 571].

By using Theorem 1.1, we can show the following.

**Lemma 2.3.** Let \( \gamma, \beta \) be real numbers, \( \beta \neq 0 \), and \( 1 > \gamma/\beta \). Let \( q(z) \in \mathcal{A} \) be univalent in \( \Delta \) and let it satisfy
\[ \Re \left( 1 + \frac{z\gamma q''(z)}{q'(z)} \right) > \begin{cases} \frac{\gamma}{\beta} & \text{if } \frac{\gamma}{\beta} \geq 0, \\ 0 & \text{if } \frac{\gamma}{\beta} \leq 0. \end{cases} \] (2.17)

If \( p(z) \in \mathcal{A} \) satisfies
\[ \gamma p(z) - \beta z p'(z) < \gamma q(z) - \beta z q'(z), \] (2.18)
then \( p(z) < q(z) \) and \( q(z) \) is the best dominant.

By using Lemma 2.3, or from Theorem 2.1, we have the following.
COROLLARY 2.4. Let $\alpha, \beta, \gamma$ be real numbers, $\beta \neq 0$, and $1 > \gamma/\beta$. Let $q(z) \in \mathcal{A}$ be univalent in $\Delta$ and let it satisfy (2.17). If $f(z) \in \mathcal{A}_0$ satisfies
\[
\frac{f(z)}{zf'(z)} \left\{ \frac{\alpha zf''(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)}\right) + \gamma \right\} < \alpha + \beta - \beta q'(z) + yq(z),
\]
then (2.15) holds and $q(z)$ is the best dominant.

By using Theorem 1.1, we obtain the following.

THEOREM 2.5. Let $\alpha \neq -1$. Let $\alpha, \beta, \gamma, \text{ and } \delta$ be real numbers, $\alpha \neq 0$, and $1 + \delta(a + 1)(\alpha + \gamma)/\alpha > 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\Delta$ and let it satisfy the following condition for $z \in \Delta$:
\[
\Re \left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} 
\frac{-\delta(a + 1)(\alpha + \gamma)}{\alpha} & \text{if } \frac{\delta(a + 1)(\alpha + \gamma)}{\alpha} \leq 0, \\
0 & \text{if } \frac{\delta(a + 1)(\alpha + \gamma)}{\alpha} \geq 0.
\end{cases}
\]

If $f(z) \in \mathcal{A}_0$ and
\[
\left\{ \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \beta \left( \frac{z}{L(a+1,c)f(z)} \right)^{\delta} + \gamma \right\} \left( \frac{L(a+1,c)f(z)}{z} \right)^{\delta} < \frac{\alpha}{\delta(a+1)} zq'(z) + (\alpha + \gamma)q(z) + \beta,
\]
then
\[
\left( \frac{L(a+1,c)f(z)}{z} \right)^{\delta} < q(z)
\]
and $q(z)$ is the best dominant.

PROOF. Define the function $p(z)$ by
\[
p(z) := \left( \frac{L(a+1,c)f(z)}{z} \right)^{\delta}.
\]
Then, clearly, $p(z)$ is analytic in $\Delta$. Also, by a simple computation, we find from (2.23) that
\[
\frac{zp'(z)}{p(z)} = \frac{\delta z(L(a+1,c)f(z))'}{L(a+1,c)f(z)} - \delta.
\]
By making use of the familiar identity (1.7) in (2.24), we get
\[
\frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} = \frac{1}{\delta(a+1)} \frac{zp'(z)}{p(z)} + 1.
\]
By using (2.23) and (2.25), we obtain
\begin{align*}
\left\{ \alpha \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} + \beta \left( \frac{z}{L(a+1,c)f(z)} \right)^{\delta} + \gamma \right\} \left( \frac{L(a+1,c)f(z)}{z} \right)^{\delta} \\
= \left\{ \alpha \left( \frac{z p'(z)}{p(z)} + 1 \right) + \beta p(z) + \gamma \right\} p(z) \\
= \frac{\alpha}{\delta(a+1)} z p'(z) + (\alpha + \gamma) p(z) + \beta.
\end{align*}
(2.26)

In view of (2.26), the subordination (2.21) becomes
\begin{equation}
\delta(a+1)(\alpha + \gamma)p(z) + \alpha z p'(z) \prec \delta(a+1)(\alpha + \gamma)q(z) + \alpha z q'(z)
\end{equation}
(2.27)
and this can be written as (1.19), where
\begin{align*}
\vartheta(w) := \delta(a+1)(\alpha + \gamma)w, \quad \varphi(w) := \alpha.
\end{align*}
(2.28)

Note that \( \varphi(w) \neq 0 \) and \( \vartheta(w), \varphi(w) \) are analytic in \( \mathbb{C} \). Let the functions \( Q(z) \) and \( h(z) \) be defined by
\begin{align*}
Q(z) : = z q'(z) \varphi(q(z)) = \alpha z q'(z), \\
h(z) : = \vartheta(q(z)) + Q(z) = \delta(a+1)(\alpha + \gamma)q(z) + \alpha z q'(z).
\end{align*}
(2.29)

By hypothesis (2.20) stated in Theorem 2.5, we see that \( Q(z) \) is starlike and
\begin{equation}
\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\delta(a+1)(\alpha + \gamma)}{\alpha} + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0.
\end{equation}
(2.30)

Thus, by an application of Theorem 1.1, the proof of Theorem 2.5 is completed.
\( \square \)

By taking \( a = c = 1 \) in Theorem 2.5 and after a suitable change in the parameters, we have the following.

**Corollary 2.6.** Let \( \alpha, \beta \neq 0 \) be real and \( 1 + \alpha > 0 \). Let \( q(z) \) be univalent in \( \Delta \) and let it satisfy (2.13). If \( f \in A_0 \) and
\begin{align*}
\left\{ \beta \frac{zf''(z)}{f'(z)} + \alpha \left( 1 - [f'(z)]^{-\beta} \right) \right\} [f'(z)]^{\beta} < z q'(z) + \alpha q(z) - \alpha,
\end{align*}
(2.31)
then
\begin{equation}
[f'(z)]^{\beta} < q(z)
\end{equation}
(2.32)
and \( q(z) \) is the best dominant.
If we take (2.16) in Corollary 2.6, we obtain a recent result of Singh [7, Theorem 1(ii), page 571].

**Theorem 2.7.** Let \( a \neq -1 \). Let \( \alpha, \beta, \) and \( \gamma \) be real numbers and let \( \beta, \gamma \neq 0 \) and \( 1 + \alpha/\gamma > 0 \). Let \( q(z) \in \mathcal{A} \) be univalent in \( \Delta \) and let it satisfy the following condition for \( z \in \Delta \):

\[
    \Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} 
        -\frac{\alpha}{\gamma} & \text{if } \frac{\alpha}{\gamma} \leq 0, \\
        0 & \text{if } \frac{\alpha}{\gamma} > 0.
    \end{cases}
\]

(2.33)

If \( f(z) \in \mathcal{A}_0 \) and

\[
    \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^{\beta} \left\{ \beta \gamma \left[ (a+1) \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \alpha \right\} \prec \gamma zq'(z) + \alpha q(z),
\]

(2.34)

then

\[
   \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^{\beta} \prec q(z)
\]

(2.35)

and \( q(z) \) is the best dominant.

**Proof.** Define the function \( p(z) \) by

\[
p(z) := \left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^{\beta}.
\]

(2.36)

Then, clearly, \( p(z) \) is analytic in \( \Delta \). Also, by a simple computation together with the use of the familiar identity (1.7), we find from (2.36) that

\[
    \frac{1}{\beta} \frac{zp'(z)}{p(z)} = (a+1) \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1.
\]

(2.37)

Therefore, it follows from (2.36) and (2.37) that

\[
\left( \frac{L(a+1,c)f(z)}{L(a,c)f(z)} \right)^{\beta} \left\{ \beta \gamma \left[ (a+1) \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} - a \frac{L(a+1,c)f(z)}{L(a,c)f(z)} - 1 \right] + \alpha \right\} = \gamma z p'(z) + \alpha p(z).
\]

(2.38)

In view of (2.38), the subordination (2.34) becomes

\[
    \gamma z p'(z) + \alpha p(z) \prec \gamma z q'(z) + \alpha q(z)
\]

(2.39)

and this can be written as (1.19), where

\[
    \vartheta(w) := \alpha w, \quad \varphi(w) := \gamma.
\]

(2.40)
Note that $\varphi(w) \neq 0$ and $\vartheta(w)$, $\varphi(w)$ are analytic in $C$. Let the functions $Q(z)$ and $h(z)$ be defined by

$$Q(z) := z q'(z) \varphi(q(z)) = y z q'(z),$$
$$h(z) := \vartheta(q(z)) + Q(z) = \alpha q(z) + y z q'(z).$$

In light of hypothesis (2.33) stated in Theorem 2.7, we see that $Q(z)$ is starlike and

$$\Re\left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re\left\{ \frac{\alpha}{y} + \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right\} > 0. \tag{2.42}$$

Since $\vartheta$ and $\varphi$ satisfy the conditions of Theorem 1.1, the result follows by an application of Theorem 1.1.

By taking $a = c = 1$ in Theorem 2.7 and after a suitable change in the parameters, we have the following.

**Corollary 2.8.** Let $\alpha, \beta \neq 0$ and $\gamma$ be real with $1 + \alpha/\gamma > 0$. Let $q(z)$ be univalent in $\Delta$ and let it satisfy (2.33).

If $f \in A_0$ and

$$\left( \frac{zf'(z)}{f(z)} \right)^{\beta} \{ \beta y \left[ 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right] + \alpha \} \prec y z q'(z) + \alpha q(z), \tag{2.43}$$

then

$$\left( \frac{zf'(z)}{f(z)} \right)^{\beta} \prec q(z) \tag{2.44}$$

and $q(z)$ is the best dominant.

If we take (2.16) and $\gamma = 1$ in Corollary 2.8, we obtain a recent result of Singh [7, Theorem 1(iii), page 571] and, by setting

$$q(z) = \int_0^1 \frac{1 - \lambda z t^\alpha}{1 + \lambda z t^\alpha} d\lambda \tag{2.45}$$

and $\alpha = 1$ in Corollary 2.8, we obtain another recent result of Singh [7, Theorem 3, page 573].

**Theorem 2.9.** Let $\alpha \neq 0$ and $\gamma$ be real numbers, $(a + 1)\alpha \gamma < 0$. Let $q(z) \in \mathcal{A}$ be univalent in $\Delta$ and let it satisfy the following condition for $z \in \Delta$:

$$\Re\left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} \frac{\alpha + \gamma(a + 1)}{\alpha} & \text{if } \frac{\alpha + \gamma(a + 1)}{\alpha} \geq 0, \\ 0 & \text{if } \frac{\alpha + \gamma(a + 1)}{\alpha} \leq 0. \end{cases} \tag{2.46}$$
If \( f(z) \in \mathcal{A}_0 \) and
\[
\frac{\alpha L(a,c)f(z)}{L(a+1,c)f(z)} \left( \frac{L(a+2,c)f(z)}{L(a+1,c)f(z)} \right) + \gamma \frac{L(a,c)f(z)}{L(a+1,c)f(z)} < \frac{a\alpha}{a+1} + \left( \frac{\alpha}{a+1} + \gamma \right) q(z) - \frac{\alpha}{a+1} zq'(z),
\]
then
\[
\frac{L(a,c)f(z)}{L(a+1,c)f(z)} < q(z)
\]
and \( q(z) \) is the best dominant.

The proof of this theorem is similar to that of Theorem 2.1 and hence it is omitted.

By taking \( a = c = 1 \) in Theorem 2.9 and after a suitable change in the parameters, we have the following.

**Corollary 2.10.** Let \( 0 \leq \alpha \leq 1 \) and \( q(z) \) be univalent in \( \Delta \) and let them satisfy
\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} 
\alpha & \text{if } \alpha \geq 0 \\
0 & \text{if } \alpha \leq 0.
\end{cases}
\]

If \( f \in \mathcal{A}_0 \) and
\[
\frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \alpha \left( \frac{f(z)}{zf'(z)} - 1 \right) < (1 + \alpha) - \alpha q(z) - zq'(z),
\]
then (2.15) holds and \( q(z) \) is the best dominant.

Let
\[
q(z) = 1 + \frac{\lambda z}{k} \int_0^1 \frac{t^\alpha}{1 + (z/k)t} dt.
\]

After a change of variable in (2.51), we get
\[
q(z) = 1 + \frac{\lambda}{z^\alpha} \int_0^z \frac{\eta^\alpha}{k + \eta} d\eta.
\]

By differentiating (2.52), we have
\[
Hzq'(z) = \frac{\lambda z}{k + z} - \alpha q(z) + \alpha
\]
or
\[
\alpha - \alpha q(z) - zq'(z) = -\frac{\lambda z}{k + z}.
\]

Since the bilinear transform
\[
w = -\frac{\lambda z}{k + z}
\]
maps $\Delta$ onto the disk
\[
\left| w + \frac{\lambda}{1 - k^2} \right| \leq \frac{\lambda|k|}{k^2 - 1},
\]
from Corollary 2.10 for the function $q(z)$ given by (2.51), we obtain a recent result of Singh [7, Theorem 2(i), page 572].

**Theorem 2.11.** Let $\alpha \neq 0$ and $\gamma$ be real numbers, $(a + 1)\alpha\gamma < 0$. Let $q(z) \in A$ be univalent in $\Delta$ and let it satisfy (2.46) for $z \in \Delta$.

If $f(z) \in A_0$ and
\[
\alpha z - \frac{L(a + 2, c)f(z)}{L(a + 1, c)f(z)} + \gamma \frac{z}{L(a + 1, c)f(z)} < (\alpha + \gamma)q(z) - \frac{\alpha}{a + 1} zf'(z),
\]
then
\[
\frac{z}{L(a + 1, c)f(z)} < q(z)
\]
and $q(z)$ is the best dominant.

The proof of this theorem is similar to that of Theorem 2.1 and therefore it is omitted. By taking $a = c = 1$ in Theorem 2.11 and after a suitable change in the parameters, we have the following.

**Corollary 2.12.** Let $0 \leq \alpha \leq 1$ and $q(z)$ be univalent in $\Delta$ and let them satisfy (2.49). If $f \in A_0$, $f(z)f'(z)/z \neq 0$, and
\[
\frac{zf''(z)}{f'(z)^2} - \alpha \left( \frac{1}{f'(z)} - 1 \right) < \alpha - \alpha q(z) - zf'(z),
\]
then
\[
\frac{1}{f'(z)} < q(z)
\]
and $q(z)$ is the best dominant.

On setting (2.51) in Corollary 2.12, we obtain a recent result of Singh [7, Theorem 2(ii), page 572].

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**References**


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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