INCLUSION RESULTS FOR CONVOLUTION SUBMETHODS

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If \( B \) is a summability matrix, then the submethod \( B_\lambda \) is the matrix obtained by deleting a set of rows from the matrix \( B \). Comparisons between Euler-Knopp submethods and the Borel summability method are made. Also, an equivalence result for convolution submethods is established. This result will necessarily apply to the submethods of the Euler-Knopp, Taylor, Meyer-König, and Borel matrix summability methods.

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1. Introduction and notation. Let \( E \) be an infinite subset of \( \mathbb{N} \cup \{0\} \) and consider \( E \) as the range of a strictly increasing sequence of nonnegative integers, say \( E := \{\lambda(n)\}_{n=0}^\infty \).

If \( B := (b_{n,k}) \) is a summability matrix, then the submethod \( B_\lambda \) is the matrix whose \( nk \)th entry is \( B_\lambda[n,k] := b_{\lambda(n),k} \). Thus, for a given sequence \( x \), the \( B_\lambda \)-transform of \( x \) is the sequence \( B_\lambda x \) with

\[
(B_\lambda x)_n = (Bx)_{\lambda(n)} := \sum_{k=0}^{\infty} b_{\lambda(n),k} x_k.
\]

(1.1)

Since \( B_\lambda \) is a row submatrix of \( B \), it is regular (i.e., limit preserving) whenever \( B \) is regular.

Row submatrices have appeared throughout the literature [5, 6, 8, 12], but they were first studied as a class unto themselves by Goffman and Petersen [7], and later by Steele [14]. The class of Cesàro submethods has been studied by Armitage and Maddox [1] and Osikiewicz [11].

Let \( A \) and \( B \) be two summability matrices. If every sequence which is \( A \)-summable is also \( B \)-summable to the same limit, then \( B \) includes \( A \), denoted by \( A \subseteq B \). Also, \( B \) is called a triangle if \( b_{n,k} = 0 \) for all \( k > n \) and \( b_{n,n} \neq 0 \) for all \( n \). The following lemma extends [1, Theorem 1].

**Lemma 1.1.** Let \( B \) be a summability matrix and let \( E := \{\lambda(n)\} \) and \( F := \{\rho(n)\} \) be infinite subsets of \( \mathbb{N} \cup \{0\} \).

1. If \( F \setminus E \) is finite, then \( B_\lambda \subseteq B_\rho \).
2. If \( B \) is a triangle and \( B_\lambda \subseteq B_\rho \), then \( F \setminus E \) is finite.
3. If \( B \) is a triangle, then \( B_\lambda \) is equivalent to \( B_\rho \) if and only if the symmetric difference \( E \triangle F \) is finite.

In particular, \( B \subseteq B_\lambda \) for any \( \lambda \).

**Proof.** Assume \( F \setminus E \) is finite and let \( x \) be a sequence that is \( B_\lambda \)-summable to \( L \). Then there exists an \( N \) such that \( \{\rho(n) : n \geq N\} \subseteq E \). That is, \( \{\rho(n) : n \geq N\} \) is a
subsequence of \{\lambda(n)\}. Since \lim_n(B\lambda x)_n = \lim_n(Bx)_{\lambda(n)} = L, we have \lim_n(B\rho x)_n = \lim_n(Bx)_{\rho(n)} = L.

Now assume \(B\) is a triangle, and hence invertible, and \(F \setminus E\) is infinite. Let \(F \setminus E := \{\rho(n(j))\}_{j=0}^{\infty}\) with \(\rho(n(j)) < \rho(n(j + 1))\). Consider the sequence \(y\) defined by

\[ y_k := \begin{cases} (-1)^j, & \text{if } k = \rho(n(j)) \text{ for some } j, \\ 0, & \text{otherwise}, \end{cases} \quad (1.2) \]

and let \(x\) be the sequence \(B^{-1}y\). Then, for every \(n\),

\[ (B\lambda x)_n = (Bx)_{\lambda(n)} = (B(B^{-1}y))_{\lambda(n)} = y_{\lambda(n)} = 0. \quad (1.3) \]

Hence, \(\lim_n(B\lambda x)_n = 0\). However, for every \(j\),

\[ (B\rho x)_{n(j)} = (Bx)_{\rho(n(j))} = (B(B^{-1}y))_{\rho(n(j))} = y_{\rho(n(j))} = (-1)^j. \quad (1.4) \]

Thus \(x\) is not \(B_{\rho}\)-summable. Therefore \(B_{\rho}\) does not include \(B_{\lambda}\), which completes the contrapositive of assertion (2). Lastly, assertion (3) follows from (1) and (2) since \(E \triangle F := (E \setminus F) \cup (F \setminus E)\).

To show the reason for the necessity of \(B\) being a triangle in assertion (2) of Lemma 1.1, consider the matrix \(B\) whose \(nk\)th entry is

\[ B[n,k] := \begin{cases} 0, & \text{if } n \text{ even and } k \neq n/2, \\ 1, & \text{if } n \text{ even and } k = n/2, \\ 0, & \text{if } n \text{ odd and } n \neq k, \\ 1, & \text{if } n \text{ odd and } n = k. \end{cases} \quad (1.5) \]

Then if \(\lambda(n) := 2n\) and \(\rho(n) := 2n + 1\), \(F \setminus E\) is infinite and \(B_{\lambda} \subseteq B_{\rho}\).

2. Inclusion results for Euler-Knopp submethods. For \(r \in \mathbb{C} \setminus \{0, 1\}\), the Euler-Knopp method of order \(r\) is given by the matrix \(E_r\) whose \(nk\)th entry is

\[ E_r[n,k] := \begin{cases} \binom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \quad (2.1) \]

For the case \(r = 1\), \(E_1\) is the identity matrix, and \(E_0\) is the matrix whose \(nk\)th entry is

\[ E_0[n,k] := \begin{cases} 1, & \text{if } k = 0, n = 0, 1, 2, \ldots, \\ 0, & \text{otherwise}. \end{cases} \quad (2.2) \]

It is well known that \(E_r\) is regular if and only if \(0 < r \leq 1\) (see [4]).
Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r \in \mathbb{C} \setminus \{0, 1\}$. The submethod $E_{r, \lambda}$ is the matrix whose $nk$th entry is

$$
E_{r, \lambda}[n, k] := \begin{cases} 
\binom{\lambda(n)}{k} r^k (1 - r)^{\lambda(n) - k}, & \text{if } k \leq \lambda(n), \\
0, & \text{if } k > \lambda(n).
\end{cases} \tag{2.3}
$$

Then $E_{r, \lambda}$ is regular if and only if $E_r$ is regular.

By a direct application of Lemma 1.1, we have the following inclusion result for the $E_{r, \lambda}$ methods.

**Lemma 2.1.** Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$ and $r \neq 0$.

(1) The method $E_{r, \lambda} \subseteq E_{r, \rho}$ if and only if $F \setminus E$ is finite.

(2) The method $E_{r, \lambda}$ is equivalent to $E_{r, \rho}$ if and only if the symmetric difference $E \Delta F$ is finite.

We now examine the relationship between $E_{r, \lambda}$ and the Borel summability method. Recall that a sequence $x$ is Borel summable to $L$ if

$$
\lim_{t \to -\infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = L. \tag{2.4}
$$

**Theorem 2.2.** Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r > 0$. Then the Borel summability method includes $E_{r, \lambda}$ if and only if $S := (\mathbb{N} \cup \{0\}) \setminus E$ is finite.

**Proof.** If $S$ is finite, then by Lemma 2.1, $E_r$ and $E_{r, \lambda}$ are equivalent. But the Borel summability method includes $E_r$ for $r > 0$ (see [4]). Hence, it also includes $E_{r, \lambda}$. If $S$ is infinite, then it may be written as a strictly increasing sequence of nonnegative integers, say $S := \{\rho(m)\}_{m=0}^{\infty}$. If $M_n := \max_{0 \leq k \leq n} |E_r[n, k]|$, consider the sequence $y$ defined by

$$
y_n := \begin{cases} 
(\rho(m)!)^2 (\rho(m) + 1) M_{\rho(m)}, & \text{if } n = \rho(m), \\
0, & \text{otherwise},
\end{cases} \tag{2.5}
$$

and let $x$ be the sequence $E_r^{-1} y$; that is, $y = E_r x$ and

$$
\lim_{n \to -\infty} (E_{r, \lambda} x)_n = \lim_{n \to -\infty} (E_r x)_{\lambda(n)} = \lim_{n \to -\infty} y_{\lambda(n)} = 0. \tag{2.6}
$$

Hence, $x$ is $E_{r, \lambda}$-summable to 0. Now observe that for a given $n$,

$$
|y_n| = |(E_r x)_n| \leq \sum_{k=0}^{n} |E_r[n, k]| |x_k| \leq M_n \sum_{k=0}^{n} |x_k|. \tag{2.7}
$$
Thus, for $n = \rho(m)$, we have

$$
(\rho(m)!)^{1/\rho(m)} \leq \left( \frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m) + 1} \cdot \frac{\gamma_{\rho(m)}}{M_{\rho(m)}} \right)^{1/\rho(m)}
$$

(2.8)

Since $\limsup_m (\rho(m)!)^{1/\rho(m)} = \infty$,

$$
\limsup_{m \to \infty} \left( \frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m) + 1} \sum_{k=0}^{\rho(m)} |x_k| \right)^{1/\rho(m)} = \infty,
$$

(2.9)

and it follows that $\limsup_n (|x_n|/n!)^{1/n} = \infty$. Thus, $\sum_{k=0}^{\infty} (x_k/k)! t^k$ diverges for all nonzero $t$ and hence $x$ is not Borel summable.

**Theorem 2.3.** There exists a sequence which is Borel summable but not $E_{r,\lambda,}$-summable for any $\lambda$ and $r > 0$.

**Proof.** Let $r > 0$ and consider the sequence $x$ defined by

$$
x_n := n \left( -\frac{1}{r} \right) \left( 1 - \frac{2}{r} \right)^{n-1}.
$$

(2.10)

Then it can be shown that $(E_{r,\lambda} x)_n = (-1)^{\lambda(n)} \lambda(n)$. Hence $x$ is not $E_{r,\lambda}$-summable for any $\lambda$. However,

$$
e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = e^{-t} \sum_{k=1}^{\infty} \left[ k \left( -\frac{1}{r} \right) \left( 1 - \frac{2}{r} \right)^{k-1} \right] \frac{t^k}{k!}
$$

$$
= \left( -\frac{1}{r} \right) e^{-t} \sum_{k=1}^{\infty} \left( 1 - \frac{2}{r} \right)^{k-1} \frac{t^k}{(k-1)!}
$$

$$
= \left( -\frac{1}{r} \right) t e^{-t} \sum_{k=0}^{\infty} \left( 1 - \frac{2}{r} \right)^{k} \frac{t^k}{k!}
$$

(2.11)

$$
= \left( -\frac{1}{r} \right) t e^{-t} e^{(1-2/r)t}
$$

$$
= \left( -\frac{1}{r} \right) t e^{-(2/r)t}.
$$

Since $r > 0$,

$$
\lim_{t \to \infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = \lim_{t \to \infty} \left( -\frac{1}{r} \right) t e^{-(2/r)t} = 0,
$$

(2.12)

and hence $x$ is Borel summable to 0. □
3. Convolution methods. Let \( p \) and \( q \) be sequences of real numbers with \( p_k \geq 0 \), \( q_k \geq 0 \), \( \sum_{k=0}^{\infty} p_k = 1 \), and \( \sum_{k=0}^{\infty} q_k = 1 \). The convolution summability method is given by the matrix \( C^* := (c_{n,k}) \) whose \( n \)th entry is

\[
c_{n,k} := \begin{cases} 
q_k, & \text{if } n = 0, \\
\sum_{j=0}^{k} c_{n-1,j} p_{k-j}, & \text{if } n \geq 1.
\end{cases}
\]  

(3.1)

It is clear that \( C^* \) is a nonnegative matrix such that for every \( n \), \( \sum_{k=0}^{\infty} c_{n,k} = 1 \). Some classical summability matrices are examples of the matrix \( C^* \). If \( 0 \leq r < 1 \), \( p := \{1-r, r, 0, 0, \ldots\} \), and \( q := \{1, 0, 0, \ldots\} \), then \( C^* \) is the Euler-Knopp method of order \( r \). If \( 0 \leq r < 1 \) and \( p := q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\} \), then \( C^* \) is the Taylor method of order \( r \), denoted by \( T_r \). If \( 0 < r < 1 \) and \( p := q := \{(1-r), (1-r)r, (1-r)r^2, \ldots\} \), then \( C^* \) is the Meyer-König method of order \( r \), denoted by \( S_r \). If \( p := q := \{1/k!\} \), then \( C^* \) is the Borel matrix method \( B^* \). Similar forms of the convolution method are known by different names, such as the random-walk method and Sonnenschein method. (Further information on all of these methods may be found in \([3, 4, 13]\).)

If \( C^* \) is the convolution method formed from the sequences \( p \) and \( q \), then let

\[
\mu := \sum_{j=0}^{\infty} j p_j, \quad \nu := \sum_{j=0}^{\infty} j q_j.
\]  

(3.2)

We note here that for the remainder of this work, \( p \) and \( q \) are nonnegative sequences whose sums are 1, and \( \mu \) and \( \nu \) represent the sums in (3.2). Also, \( c_{n,k} := 0 \) whenever \( k < 0 \).

We next present some preliminary results concerning the convolution method.

**Lemma 3.1.** The convolution method \( C^* \) is regular if and only if \( p_0 < 1 \).

**Proof.** See [9].

**Lemma 3.2.** If \( \mu < \infty \) and \( \nu < \infty \), then for every \( n \),

\[
\sum_{k=0}^{\infty} k c_{n,k} = n \mu + \nu.
\]  

(3.3)

**Proof.** Note that for \( n = 0 \), the result holds. So assume the result holds for some integer \( n > 0 \). Then

\[
\sum_{k=0}^{\infty} k c_{n+1,k} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} c_{n,j} p_{k-j} \right) = \sum_{j=0}^{\infty} c_{n,j} \sum_{k=j}^{\infty} k p_{k-j} = \sum_{j=0}^{\infty} c_{n,j} \left( \sum_{i=0}^{\infty} i p_i + \sum_{i=0}^{\infty} p_i \right) = \sum_{j=0}^{\infty} j c_{n,j} = (n+1)\mu + \nu.
\]  

(3.4)

By induction, the result follows.
**Lemma 3.3.** Let $C^*$ be the convolution method formed from the sequences $p$ and $q$ and $D^* := (d_{n,k})$ the convolution method formed from the sequences $p$ and $\tilde{q} := \{1,0,0,\ldots\}$. Then for nonnegative integers $n$, $k$, and $j$,

$$c_{n+j,k} = \sum_{i=0}^{k} c_{n,k-i} d_{j,i}. \quad (3.5)$$

The proof of this lemma is a straightforward induction argument left to the reader.

**Lemma 3.4.** Let $C^*$ be the convolution method formed from the sequences $p$ and $q$. If $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$, then

$$\sum_{k=0}^{\infty} \left| c_{n,k+1} - c_{n,k} \right| = O \left( \frac{1}{\sqrt{n}} \right). \quad (3.6)$$

**Proof.** Let $D^* := (d_{n,k})$ be the convolution method formed from the sequences $p$ and $\tilde{q} := \{1,0,0,\ldots\}$. We first prove that the result holds for $D^*$.

Let $\phi(t) := (\sqrt{2\pi} e^{t^2/2})^{-1}$ and $x_{n,k} := (k - n\mu)/\sigma \sqrt{n}$, where $\sigma^2 := \sum_{j=0}^{\infty} (j - \mu)^2 p_j$. Then

$$\sqrt{n} \sum_{k=0}^{\infty} \left| d_{n,k+1} - d_{n,k} \right| \leq \sqrt{n} \sum_{k=0}^{\infty} \left| d_{n,k+1} - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) \right|$$

$$+ \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) \right|$$

$$+ \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) - d_{n,k} \right|. \quad (3.7)$$

The first and the third terms on the right-hand side of the inequality are bounded by a result of Bikjalis and Jasjunas [2]. For the middle term, the mean value theorem yields

$$\sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma \sqrt{n}} \phi(x_{n,k}) \right| = \frac{1}{\sigma} \sum_{k=0}^{\infty} \left| \phi'(\xi_{n,k}) \right| (x_{n,k+1} - x_{n,k})$$

$$< K \frac{1}{\sigma} \int_{\mathbb{R}} |\phi'(t)| \, dt < \infty, \quad (3.8)$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$ and $K > 0$ is some constant. Thus, the result holds for the convolution method $D^*$. Then, by Lemma 3.3,
\[\begin{align*}
\leq & \sum_{k=0}^{\infty} q_{k+1} + \sum_{k=0}^{\infty} q_{k-i} \left| d_{n,i+1} - d_{n,i} \right| \\
\leq & \sum_{i=0}^{\infty} \left| d_{n,i+1} - d_{n,i} \right| \sum_{k=i}^{\infty} q_k \\
= & \sum_{i=0}^{\infty} \left| d_{n,i+1} - d_{n,i} \right| \left( \sum_{k=i}^{\infty} q_k \right) = O\left( \frac{1}{\sqrt{n}} \right).
\end{align*}\]

(3.9)

4. Equivalence results for convolution submethods. Let \( E := \{ \lambda(n) \} \) be an infinite subset of \( \mathbb{N} \cup \{0\} \). The convolution submethod \( C^*_\lambda \) is the matrix whose \( nk \)th entry is

\[ C^*_\lambda[n,k] := C^*[\lambda(n),k]. \] (4.1)

**Lemma 4.1.** The convolution submethod \( C^*_\lambda \) is regular if and only if \( p_0 < 1 \).

**Proof.** If \( p_0 < 1 \), then \( C^* \) is regular and hence \( C^*_\lambda \) is also regular. Conversely, if \( C^*_\lambda \) is regular and \( p_0 = 1 \), then there exists a \( \hat{k} \) such that \( q_{\hat{k}} \neq 0 \). Then \( \lim_{n} C^*_\lambda[n,\hat{k}] = q_{\hat{k}} \neq 0 \), which contradicts the regularity of \( C^*_\lambda \).

The following theorem compares \( C^*_\lambda \) with \( C^* \) for bounded sequences.

**Theorem 4.2.** Let \( C^* \) be the convolution method formed from the sequences \( p \) and \( q \) with \( \mu < \infty \), \( \nu < \infty \), \( 0 < \sum_{j=0}^{\infty} (j-\mu)^2 p_j \), and \( \sum_{j=0}^{\infty} j^3 p_j < \infty \). Let \( E := \{ \lambda(n) \} \) be an infinite subset of \( \mathbb{N} \cup \{0\} \). If

\[ \lim_{n \to \infty} \frac{\lambda(n+1) - \lambda(n)}{\sqrt{\lambda(n)}} = 0, \] (4.2)

then \( C^* \) and \( C^*_\lambda \) are equivalent for bounded sequences.

**Proof.** By Lemma 1.1, \( C^* \subseteq C^*_\lambda \) for any \( \lambda \). So assume \( \lim_n (\lambda(n+1) - \lambda(n)) / \sqrt{\lambda(n)} = 0 \) and let \( x \) be a bounded sequence that is \( C^*_\lambda \)-summable to \( L \). Consider the set \( S := \{ \rho(n) \} := (\mathbb{N} \cup \{0\}) \setminus E \). If \( S \) is finite, then Lemma 1.1 shows that \( C^*_\lambda \) and \( C^* \) are equivalent for all sequences. So assume \( S \) is infinite. Then there exists an \( N \) such that for \( n \geq N \), \( \rho(n) > \lambda(0) \). Since \( E \) and \( S \) are disjoint, for \( n \geq N \), there exists an integer \( m \) such that \( \lambda(m) < \rho(n) < \lambda(m+1) \). We write \( \rho(n) := \lambda(m) + j \), where \( 0 < j < \lambda(m+1) - \lambda(m) \). Then, for \( n \geq N \),

\[ \left| (C^*_\rho x)_n - (C^*_\lambda x)_m \right| = \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m),k} x_k \right| = \left| \sum_{k=0}^{\infty} c_{\lambda(m)+j,k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m),k} x_k \right|. \] (4.3)
By Lemma 3.3, this becomes

$$\left| (C^*_\rho x)_n - (C^*_\lambda x)_m \right| = \left| \sum_{k=0}^{\infty} \left( \sum_{i=0}^{\infty} c_{\lambda(m), k-i} d_{j,i} \right) x_k - \sum_{k=0}^{\infty} c_{\lambda(m), k} x_k \right|$$

$$= \left| \sum_{k=0}^{\infty} x_k \left( \sum_{i=0}^{\infty} c_{\lambda(m), k-i} d_{j,i} \right) - \left( \sum_{i=0}^{\infty} c_{\lambda(m), k} d_{j,i} \right) \right|$$

$$\leq \|x\|_\infty \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} |c_{\lambda(m), k-i} - c_{\lambda(m), k}||d_{j,i}|$$

$$= \|x\|_\infty \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |i-1| \left| \sum_{i=0}^{\infty} c_{\lambda(m), k-i} - c_{\lambda(m), k-i-1} \right|$$

$$\leq \|x\|_\infty \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{i-1} |c_{\lambda(m), k-i} - c_{\lambda(m), k-i-1}||d_{j,i}|$$

$$= \|x\|_\infty \sqrt{\lambda(m)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{i-1} \sqrt{l} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}||d_{j,i}|.$$  \(4.4\)

By Lemma 3.4, there exists an \(M > 0\) such that

$$\sqrt{\lambda(m)} \sum_{k=0}^{\infty} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}| < M.  \tag{4.5}$$

Then, by Lemma 3.2,

$$\left| (C^*_\rho x)_n - (C^*_\lambda x)_m \right| \leq \|x\|_\infty \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{i-1} |d_{j,i}| \frac{\|x\|_\infty M}{\sqrt{\lambda(m)}} \sum_{k=0}^{\infty} \sqrt{i} \sum_{l=0}^{i-1} \sqrt{l}$$

$$\leq \|x\|_\infty \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{i-1} \sqrt{l} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}||d_{j,i}| \leq \|x\|_\infty M \cdot j\mu.  \tag{4.6}$$

Since \(0 < j < \lambda(m+1) - \lambda(m)\),

$$\left| (C^*_\rho x)_n - (C^*_\lambda x)_m \right| < \|x\|_\infty M \mu \cdot \frac{\lambda(m+1) - \lambda(m)}{\sqrt{\lambda(m)}} = o(1).  \tag{4.7}$$

Thus,

$$0 \leq \left| (C^*_\rho x)_n - L \right| \leq \left| (C^*_\rho x)_n - (C^*_\lambda x)_m \right| + \left| (C^*_\lambda x)_m - L \right| = o(1) + o(1) = o(1).  \tag{4.8}$$

Therefore, the sequence \(C^*x\) may be partitioned into two disjoint subsequences, namely \((C^*_\rho x)_n = (C^*x)_{\lambda(n)}\) and \((C^*_\lambda x)_n = (C^*x)_{\rho(n)}\), each having the common limit \(L\). Thus, \(x\) must be \(C^*\)-summable to \(L\), and hence \(C^*\) and \(C^*_\lambda\) are equivalent for bounded sequences.

The following theorem is a well-known result due to Meyer-König (see [10, Theorem 25]).

**Theorem 4.3.** The methods \(E_r\) \((0 < r < 1)\), \(S_r\) \((0 < r < 1)\), \(T_r\) \((0 < r < 1)\), and the Borel method are equivalent for bounded sequences.
Since the Euler-Knopp methods of order $0 < r < 1$, Taylor methods of order $0 < r < 1$, Meyer-König methods of order $0 < r < 1$, and the Borel matrix method all have generating sequences satisfying the conditions in Theorem 4.2, the following corollary is immediate.

**Corollary 4.4.** Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $0 < r < 1$. If $\lambda$ satisfies condition (4.6), then $E_{r,\lambda}, E_r, T_{r,\lambda}, T_r, S_{r,\lambda}, S_r, B_{\lambda}^*, B^*$, and the Borel method are all equivalent for bounded sequences.

The next theorem presents an equivalence relationship between the $C^*_\lambda$ submethods.

**Theorem 4.5.** Let $C^*$ be the convolution method formed from the sequences $p$ and $q$ with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$. If

$$\lim_{n \to \infty} \frac{\rho(n) - \lambda(n)}{\sqrt{\lambda(n)}} = 0,$$

then $C^*_\lambda$ and $C^*_\rho$ are equivalent for bounded sequences.

**Proof.** Let $x$ be a bounded sequence and consider the sequences $M(n) := \max\{\lambda(n), \rho(n)\}$ and $m(n) := \min\{\lambda(n), \rho(n)\}$. We write $M(n) := m(n) + j$, where $j := M(n) - m(n)$. For $n \geq 1$, we have

$$\left| (C^*_\rho x)_n - (C^*_\lambda x)_n \right| = \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(n),k} x_k \right| = \left| \sum_{k=0}^{\infty} c_{M(n),k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right| = \left| \sum_{k=0}^{\infty} c_{m(n)+j,k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right|. \quad (4.10)$$

Then, as in the proof of Theorem 4.2, we have

$$\left| (C^*_\rho x)_n - (C^*_\lambda x)_n \right| \leq O(1) \frac{j}{\sqrt{m(n)}} = O(1) \frac{M(n) - m(n)}{\sqrt{m(n)}} = O(1) \frac{\sqrt{\lambda(n)}}{\sqrt{m(n)}} \frac{\rho(n) - \lambda(n)}{\sqrt{\lambda(n)}} = O(1) \cdot O(1) \cdot o(1) = o(1). \quad (4.11)$$

Then if $x$ is $C^*_\lambda$-summable to $L$,

$$0 \leq \left| (C^*_\rho x)_n - L \right| \leq \left| (C^*_\rho x)_n - (C^*_\lambda x)_n \right| + \left| (C^*_\lambda x)_n - L \right| = o(1) + o(1) = o(1). \quad (4.12)$$
Similarly, if \( x \) is \( C^*_\rho \)-summable to \( L \), then

\[
0 \leq |(C^*_\lambda x)_n - L| \leq |(C^*_\lambda x)_n - (C^*_\rho x)_n| + |(C^*_\rho x)_n - L| = o(1) + o(1) = o(1).
\]

Thus, \( C^*_\lambda \) and \( C^*_\rho \) are equivalent for bounded sequences.

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**References**


Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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