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FLAT SEMIMODULES

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To my dearest friend Najla Ali

We introduce and investigate flat semimodules and k-flat semimodules. We hope these concepts will have the same importance in semimodule theory as in the theory of rings and modules.

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1. Introduction. We introduce the notion of flat and k-flat. In Section 2, we study the structure ensuing from these notions. Proposition 2.4 asserts that V is flat if and only if $(V \otimes_R -)$ preserves the exactness of all right-regular short exact sequences. Proposition 2.5 gives necessary and sufficient conditions for a projective semimodule to be k-flat. In Section 3, Proposition 3.3 gives the relation between flatness and injectivity. In Section 4, Proposition 4.1 characterizes the k-flat cancellable semimodules with the left ideals. Proposition 4.4 describes the relationship between the notions of projectivity and flatness for a certain restricted class of semirings and semimodules. Throughout, R will denote a semiring with identity 1. All semimodules M will be left R-semimodules, except at cited places, and in all cases are unitary semimodules, that is, $1 \cdot m = m$ for all $m \in M$ $(m \cdot 1 = m$ for all $m \in M)$ for all left R-semimodules $_RM$ (resp., for all right R-semimodule M_R).

We recall here (cf. [1, 2, 4, 7, 8]) the following facts.

(a) A semiring R is said to satisfy the left cancellation law if and only if for all $a,b,c \in R$, $a+b = a+c \Rightarrow b = c$. A semimodule M is said to satisfy the left cancellation law if for all $m,m',m'' \in M$, $m+m' = m+m'' \Rightarrow m' = m''$.

(b) We say that a nonempty subset N of a left semimodule M is subtractive if and only if for all $m,m' \in M$, m, $m+m'$ in N imply m' in N.

(c) A semiring R is called completely subtractive if $_RR$ is a completely subtractive semimodule; and a left R-semimodule M is called completely subtractive if and only if for every subsemimodule N of M, N is subtractive.

(d) A semimodule M is said to be free R-semimodule if M has a basis over R.

(e) A semimodule C is said to be semicogenerated by U when there is a homomorphism $\varphi : M \to \Pi_A C$ such that $\ker\theta = 0$. A semimodule C is said to be a semicogenerator when C semicogenerates every left R-semimodule M.

(f) Let $\alpha : M \to N$ be a homomorphism of semimodules. The subsemimodule $\operatorname{Im}\alpha$ of N is defined as follows: $\operatorname{Im}\alpha = \{n \in N : n+\alpha(m') = \alpha(m)$ for some $m,m' \in M\}$. Also α is said to be a semimonomorphism if $\ker\alpha = 0$, to be a semi-isomorphism if α is surjective and $\operatorname{Ker}\alpha = 0$, to be an isomorphism if α is injective and surjective, to be i-regular if $\alpha(M) = \operatorname{Im}\alpha$, to be k-regular if for $a, a' \in A$, $\alpha(a) = \alpha(a')$ implying $a + k = a' + k'$ for some $k, k' \in \ker\alpha$, and to be regular if it is both i-regular and k-regular.

(g) An R-semimodule M is said to be k-regular if there exist a free R-semimodule F and a surjective R-homomorphism $\alpha : F \to M$ such that α is k-regular.

(h) The sequence $K \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is called an exact sequence if $\operatorname{Ker}\beta = \operatorname{Im}\alpha$, and proper exact if $\operatorname{Ker}\beta = \alpha(K)$.

(i) A short sequence $0 \to K \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ is said to be left k-regular right regular if α is k-regular and β is right regular.

(j) For any two R-semimodules N, M, $\operatorname{Hom}_R(N, M) := \{\alpha : N \to M \mid \alpha \text{ is an } R\text{-homomorphism of semimodules}\}$ is a semigroup under addition. If M, N, and U are R-semimodules and $\alpha : M \to N$ is a homomorphism, then $\operatorname{Hom}(\alpha, I_U) : \operatorname{Hom}_R(N, U) \to \operatorname{Hom}_R(M, U)$ is given by $\operatorname{Hom}(\alpha, I_U)\gamma = \gamma\alpha$, where I_U is the identity on U.

(k) If M is a right R-semimodule, N is a left R-semimodule, and T is an $\mathbf{N}$-semimodule, then a function $\theta : M \times N \to T$ is R-balanced if and only if, for all $m, m' \in M$, for all $n, n' \in N$, and for all $r \in R$, we have

(1) $\theta(m + m', n) = \theta(m, n) + \theta(m', n)$,

(2) $\theta(m, n + n') = \theta(m, n) + \theta(m, n')$,

(3) $\theta(mr, n) = \theta(m, rn)$.

Let R be a semiring, let M be a right R-semimodule, and let N be a left R-semimodule. Let A be the set $M \times N$, and let U be the $\mathbf{N}$-semimodule $\oplus_A \mathbf{N} \times \oplus_A \mathbf{N}$. Let W be the subset of U consisting of all elements of the following forms:

(1) $(\alpha[m + m', n], \alpha[m, n] + \alpha[m', n])$,

(2) $(\alpha[m, n] + \alpha[m', n], \alpha[m + m', n])$,

(3) $(\alpha[m, n + n'], \alpha[m, n] + \alpha[m, n'])$,

(4) $(\alpha[m, n] + \alpha[m, n'], \alpha[m, n + n'])$,

(5) $(\alpha[mr, n], \alpha[m, rn])$,

(6) $(\alpha[m, rn], \alpha[mr, n])$,

for m and m' in M, n and n' in N, and r in R, and where $\alpha[m, n]$ is the function from $M \times N$ to $\mathbf{N}$ which sends (m, n) to 1 and sends every other element of $M \times N$ to 0. Let U' be the $\mathbf{N}$-subsemimodule of U generated by W. Define $\mathbf{N}$ congruence relation $\equiv$ on $\oplus_A \mathbf{N}$ by setting $\alpha \equiv \alpha'$ if and only if there exists an element $(\beta, \gamma) \in U'$ such that $\alpha + \beta = \alpha' + \gamma$. The factor $\mathbf{N}$-semimodule $\oplus_A \mathbf{N} / \equiv$ will be denoted by $M \otimes_R N$, and is called the tensor product of M and N over R.

(A) A left R-semimodule P is said to be projective semimodule if and only if for each surjective R-homomorphism $\varphi : M \to N$, the induced homomorphism $\overline{\varphi} : \operatorname{Hom}_R(P, M) \to \operatorname{Hom}_R(P, N)$ is surjective.

2. Flat and k-flat semimodules.

In this section, we discuss the structure of flat and k-flat semimodules. Proposition 2.4 asserts that V is flat if and only if $(V \otimes_R -)$ preserves the exactness of all left k-regular right regular short sequences. In Proposition 2.5, we give the necessary and sufficient condition for the projective right semimodule to be k-flat relative to a cancellable left semimodule.

Definition 2.1. A semimodule V_R is flat relative to a semimodule ${}_RM$ (or that V is M-flat) if and only if for every subsemimodule $K \le M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is proper exact (i.e., $\operatorname{Ker}(I_V \otimes_R i_K) = 0$) where $I_V \otimes_R i_K(v \otimes k) = v \otimes i_K(k)$. A semimodule V_R that is flat relative to every left R-semimodule is called a flat right R-semimodule.

Definition 2.2. A semimodule V_R is k-flat relative to a semimodule ${}_RM$ (or that V is Mk-flat) if and only if for every subsemimodule $K \le M$, the sequence $0 \to V \otimes_R K \xrightarrow{I_V \otimes_R i_K} V \otimes_R M$ is proper exact and $I_V \otimes i_K$ is k-regular (i.e., $I_V \otimes_R i_K$ is injective). A semimodule V_R that is k-flat relative to every right R-semimodule is called a k-flat right R-semimodule. Thus, if V_R is k-flat relative to ${}_RM$, then V_R is flat relative to ${}_RM$.

Our next result shows that the class of flat and k-flat semimodules is closed under direct sums.

Proposition 2.3. *Let $(V_\alpha)_{\alpha \in A}$ be an indexed set of right R-semimodules. Then $\oplus_A V_\alpha$ is M-flat (k-flat) if and only if each V_α is M-flat (k-flat).*

Proof. Let M be a left R-semimodule and K a subsemimodule of M. Consider the following commutative diagram:

$$\begin{array}{ccc}
V_\alpha \otimes K & \xrightarrow{I_{V\alpha} \otimes i_K} & V_\alpha \otimes M \\
\pi_\alpha \uparrow \downarrow i_\alpha & & i_\alpha \downarrow \uparrow \pi_\alpha \\
\oplus(V_\alpha \otimes K) & \xrightarrow{\theta} & \oplus(V_\alpha \otimes M) \\
\varphi \updownarrow & & \updownarrow \varphi' \\
(\oplus V_\alpha) \otimes K & \xrightarrow{I_{\oplus V\alpha} \otimes i_K} & (\oplus V_\alpha) \otimes M,
\end{array} \tag{2.1}$$

where $\pi_\alpha : \oplus(V_\alpha \otimes K) \to V_\alpha \otimes K$ and $i_\alpha : V_\alpha \otimes K \to \oplus(V_\alpha \otimes K)$ are defined respectively by $\pi_\alpha : (v_\alpha \otimes k_\alpha) \hbar v_\alpha \otimes k_\alpha$ and $i_\alpha : v_\alpha \otimes k_\alpha \hbar (v_i \otimes k_i)$, where $v_i \otimes k_i = 0$ if $i \ne \alpha$ and $v_i \otimes k_i = v_\alpha \otimes k_\alpha$ if $\alpha = i$; φ and φ' are the isomorphisms of [8, Proposition 5.4] given by $\varphi[(v_\alpha) \otimes k] = (v_\alpha \otimes k)$ and $\theta(v_\alpha \otimes k) = (v_\alpha \otimes i(k))$. Now suppose that $\oplus V_\alpha$ is M-flat (k-flat). If $I_{V_\alpha} \otimes i_K(v_\alpha \otimes k) = 0[I_{V_\alpha} \otimes i_K((v_\alpha \otimes k)) = I_{V_\alpha} \otimes i_K((v'_\alpha \otimes k'))]$, then by the above diagram we have $(v_\alpha) \otimes i_K(k) = 0[(v_\alpha) \otimes i(k) = (v'_\alpha) \otimes i(k')]$. Since $\oplus V_\alpha$ is flat (k-flat), then $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v'_\alpha) \otimes k']$. Again by (2.1), $(v_\alpha \otimes k) = 0$ whence $v_\alpha \otimes k = 0[(v_\alpha \otimes k) = (v'_\alpha \otimes k')$, whence $v_\alpha \otimes k = v'_\alpha \otimes k']$. Therefore V_α is flat (k-flat).

Conversely, suppose that V_α is M-flat (k-flat) for each $\alpha \in A$. If $I_{\oplus V_\alpha} \otimes i_K((v_\alpha) \otimes k) = 0[I_{\oplus V_\alpha} \otimes i_K((v_\alpha) \otimes k) = I_{\oplus V_\alpha} \otimes i_K((v'_\alpha) \otimes k')]$, then by the above diagram we have $v_\alpha \otimes i(k) = 0[v_\alpha \otimes i(k) = v'_\alpha \otimes i(k')]$ for each $\alpha \in A$. Since V_α is flat (k-flat), then $v_\alpha \otimes k = 0[v_\alpha \otimes k = v'_\alpha \otimes k']$ for each α. Therefore, $(v_\alpha \otimes k) = 0[(v_\alpha \otimes k) = (v'_\alpha \otimes k')]$. Again by (2.1), $(v_\alpha) \otimes k = 0[(v_\alpha) \otimes k = (v'_\alpha) \otimes k']$. Thus $\oplus V_\alpha$ is flat (k-flat). □

Proposition 2.4. *Let M be a left R-semimodule. A right R-semimodule V is M flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left k-regular right regular* *short exact sequences with middle term M:*

$$0 \longrightarrow {}_RK \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \longrightarrow 0. \tag{2.2}$$

Proof. "If" part. Let $0 \to {}_RK \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \to 0$ be a left k-regular right regular exact sequence. Since V_R is ${}_RM$-flat, then using [8, Theorem 5.5(2)], the sequence

$$0 \longrightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \longrightarrow 0 \tag{2.3}$$

is exact.

"Only if" part. Let ${}_RK \le {}_RM$. Consider the following exact sequence:

$$0 \longrightarrow K \xrightarrow{i_K} M \xrightarrow{\pi_{\operatorname{Im} i_K}} M/\operatorname{Im} i_K \longrightarrow 0. \tag{2.4}$$

By hypothesis, $0 \to V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is an exact sequence. Thus V is M-flat. □

Our next result gives a necessary and sufficient condition for a projective semimodule to be k-flat relative to a cancellable semimodule M.

Proposition 2.5. *Let V_R be projective and ${}_RM$ cancellable. Then, V is Mk-flat if and only if the functor $(V \otimes_R -)$ preserves the exactness of all left k-regular right regular short exact sequences*

$$0 \longrightarrow {}_RK \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \longrightarrow 0. \tag{2.5}$$

Proof. "If" part. Let $0 \to {}_RK \xrightarrow{\alpha} {}_RM \xrightarrow{\beta} {}_RN \to 0$ be a left k-regular right regular exact sequence. Since V_R is ${}_RM$ k-flat, then V_R is ${}_RM$-flat. By using Proposition 2.4, the sequence

$$0 \longrightarrow V \otimes_R K \xrightarrow{I_V \otimes \alpha} V \otimes_R M \xrightarrow{I_V \otimes \beta} V \otimes_R N \longrightarrow 0 \tag{2.6}$$

is exact.

"Only if" part. Let $K \le M$. Consider the following exact sequence:

$$0 \longrightarrow K \xrightarrow{i_K} M \xrightarrow{\pi_{\operatorname{Im} i_K}} M/\operatorname{Im} i_K \longrightarrow 0. \tag{2.7}$$

Since V is projective and M is cancellable, then by using [9, Proposition 1.16], $I_V \otimes i_K$ is k-regular. By hypothesis, $0 \to V \otimes_R K \xrightarrow{I_V \otimes i_K} V \otimes_R M$ is an exact sequence. Thus V is Mk-flat. □

3. Flatness via injectivity.

We will discuss the relation between the injectivity and flatness. By $(\cdot)^*$ we mean the functor $\operatorname{Hom}_{\mathbf{N}}(-,C)$, where C is a fixed injective semicogenerator cancellative $\mathbf{N}$-semimodule.

Remark 3.1. If U is a right R-semimodule, then U^* is a left R-semimodule.

Proof. Let $\alpha \in \operatorname{Hom}_N(U,C)$ and let $r \in R$. Define $r\alpha(u) = \alpha(ur)$. If $s \in R$, then $s(r\alpha)u = (r\alpha)(us) = \alpha(usr) = (sr)\alpha(u)$. Therefore, U^* is a left R-semimodule. □

We state and prove the following lemma, analogous to the one on modules which is needed in the proof of Proposition 3.3.

Lemma 3.2. *Let R be a semiring, let M and M' be left R-semimodules, and let U be a right R-semimodule. Let T be a cancellative $\mathbf{N}$-semimodule. If $\alpha : M' \to M$ is an R-homomorphism, then there exist $\mathbf{N}$-isomorphisms φ and φ' such that the following diagram commutes:*

$$\begin{array}{ccc} \operatorname{Hom}_R(M,\operatorname{Hom}_{\mathbf{N}}(U,T)) & \xrightarrow{\operatorname{Hom}_R(\alpha, I_{\operatorname{Hom}_{\mathbf{N}}}(U,T))} & \operatorname{Hom}_R(M',\operatorname{Hom}_{\mathbf{N}}(U,T_N)) \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi'} \\ \operatorname{Hom}_{\mathbf{N}}((U\otimes_R M),T) & \xrightarrow{\operatorname{Hom}_N((I_U\otimes\alpha),I_T)} & \operatorname{Hom}_{\mathbf{N}}((U\otimes_R M'),T). \end{array} \tag{3.1}$$

Proof. By [7, Proposition 14.15], there exists an $\mathbf{N}$-isomorphism

$$\varphi : \operatorname{Hom}_R(M,\operatorname{Hom}_{\mathbf{N}}(U,T)) \longrightarrow \operatorname{Hom}_R(M\otimes U,T) \tag{3.2}$$

given by $\varphi(\gamma) : u\otimes m\hbar\gamma(m)u$. Then with a parallel definition for φ', we have

$$\begin{aligned} &\varphi'\hbar\operatorname{Hom}_R(\alpha, I_{\operatorname{Hom}_N(U,T)})(\gamma)(u\otimes m') \\ &\quad= \varphi'(\gamma\alpha)(u\otimes m') = (\gamma\alpha)(m')(u) \\ &\quad= \gamma(\alpha(m'))(u) = \varphi(\gamma)(u\otimes\alpha(m')) \\ &\quad= \varphi(\gamma)\hbar(I_U\otimes\alpha)(u\otimes m') \\ &\quad= \operatorname{Hom}_{\mathbf{N}}(I_U\otimes\alpha, I_T)(\varphi(\gamma))(u\otimes m'), \end{aligned} \tag{3.3}$$

and the diagram commutes. □

Proposition 3.3. *Let M be a left R-semimodule.*
(1) *If the right R-semimodule V is Mk-flat, then V^* is M-injective.*
(2) *If V^* is M-injective, then V is M-flat.*

Proof. (1) Let K be a subsemimodule of M. Since V is Mk-flat, then the sequence $0\to V\otimes K \xrightarrow{I_V\otimes i_K} V\otimes M$ is proper exact, and $I_V\otimes i_K$ is k-regular. By Lemma 3.2, we have the following commutative diagram:

$$\begin{array}{ccccc} \operatorname{Hom}_R(M,V^*) & \xrightarrow{\operatorname{Hom}_R(i_K,I_{V^*})} & \operatorname{Hom}_R(K,V^*) & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle\varphi'} & & & & \Big\downarrow{\scriptstyle\varphi} \\ (V\otimes M)^* & \xrightarrow{\operatorname{Hom}(I_V\otimes i_K,I_C)} & (V\times K)^* & \longrightarrow & 0, \end{array} \tag{3.4}$$

where φ' and φ are $\mathbf{N}$-isomorphisms. It follows that the top row is proper exact if and only if the bottom row is proper exact, whence by [6, Proposition 3.1], V^* is injective.

(2) If V^* is injective, then

$$\operatorname{Hom}(M,V^*) \xrightarrow{\operatorname{Hom}(i_K,I_{V^*})} \operatorname{Hom}(K,V^*) \longrightarrow 0 \tag{3.5}$$

is proper exact. Again by the above diagram,

$$(V\otimes M)^* \xrightarrow{\operatorname{Hom}(I_V\otimes i_K, I_c)} (V\otimes K)^* \longrightarrow 0 \tag{3.6}$$

is proper exact. Hence, the sequence is exact. Since C is a semicogenerator, then by [3, Proposition 4.1], the sequence $0\to V\otimes K\to V\otimes M$ is an exact sequence. Hence, V is M-flat. □

4. Cancellable semimodules. In this section, we deal with cancellable semimodules. We characterize k-flat cancellable semimodules by means of left ideals.

Proposition 4.1. *The following statements about a cancellable right R-semimodule V are equivalent:*

(1) *V is k-flat relative to ${}_RR$;*

(2) *for each (finitely generated) left ideal $I\le {}_RR$, the surjective* **N***-homomorphism $\varphi : V\otimes_R I\to VI$ with $\varphi(v\otimes a)=va$ is a k-regular semimonomorphism.*

Proof. (1)⇒(2). Since V is cancellable, then by using [7, Proposition 14.16], $V\otimes_R R\simeq V$. Consider the following commutative diagram:

$$\begin{array}{ccc} V\otimes_R I & \xrightarrow{I_V\otimes i_I} & V\otimes_R R \\ \varphi\big\downarrow & & \big\downarrow\theta \\ VI & \xrightarrow{i_{VI}} & V, \end{array} \tag{4.1}$$

where θ is the isomorphism of [7, Proposition 14.16]. Since $\psi : V\times I\to VI$ given by $\psi(v,i)=vi$ is an R-balanced function, then by using [7, Proposition 14.14], there is an exact unique **N**-homomorphism $\varphi : V\otimes I\to VI$ satisfying the condition $\varphi(v\otimes i)=\psi(v,i)$. Since V is k-flat relative to ${}_RR$, then $I_V\otimes_R i_I$ is injective. If $\varphi(\Sigma v_i\otimes a_i)=\varphi(\Sigma v_i'\otimes a_i')$, then $\theta(I_V\otimes_R i_I)(\Sigma v_i\otimes a_i)=\theta(I_V\otimes_R i_I)(\Sigma v_i'\otimes a_i')$. Since θ and $I_V\otimes i_I$ are injective, then $\Sigma(v_i\otimes a_i)=(\Sigma v_i'\otimes a_i')$.

(2)⇒(1). Again consider the above diagram. Let I be any left ideal of R and let $I_V\otimes_R i_I(\Sigma v_i\otimes a_i)=I_V\otimes_R i_I(\Sigma v_i'\otimes a_i')$, where $\Sigma v_i'\otimes a_i'$, $\Sigma v_i\otimes a_i\in V\otimes_R I$. Let $K_1=\Sigma Ra_i$, $K_2=\Sigma Ra_i'$, and $K=K_1+K_2$. Now $\theta(I_V\otimes i_I)(\Sigma v_i\otimes a_i)=\theta(I_V\otimes i_I)(\Sigma v_i'\otimes a_i')$, whence $\Sigma v_ia_i=\Sigma v_i'a_i'$. Now consider the following diagram, where $i_K : K\to I$ is the inclusion map:

$$\begin{array}{ccc} V\otimes K & \xrightarrow{I_V\otimes_R i_K} & V\otimes I \\ \varphi_K\big\downarrow & & \big\downarrow\theta \\ VK & \xrightarrow{i_{VK}} & V. \end{array} \tag{4.2}$$

By hypothesis, φ_K is monic. Thus, $\Sigma_i v_i\otimes a_i=\Sigma_i v_i'\otimes a_i'$ as an element of $V\otimes K$. Hence, $I_V\otimes_R i_K(\Sigma_i v\otimes a_i)=I_V\otimes i_K(\Sigma_i v_i'\otimes a_i')\in V\otimes I$, and $\Sigma v_i\otimes a_i=\Sigma v_i'\otimes a_i'$ as an element of $V\otimes I$. Therefore, $I_V\otimes_R i_I$ is monic. Hence, V is k-flat relative to ${}_RR$. □

Proposition 4.2. *Let M be a cancellable left R-semimodule. Then R_R is Mk-flat.*

Proof. Let $i_K : K \to M$ be the inclusion homomorphism. By [7, Proposition 14.16], $R \otimes_R K \simeq K$ and $R \otimes_R M \simeq M$. Consider the following commutative diagram:

$$\begin{array}{ccc} R \otimes_R K & \xrightarrow{I_R \otimes i} & R \otimes_R M \\ \big| \simeq & & \big| \simeq \\ K & \xrightarrow{i_K} & M, \end{array} \tag{4.3}$$

since i_K is injective, then $I \otimes_R i_K$ is injective. □

Corollary 4.3. *Let M be a cancellable left R-semimodule. Then every free R-semimodule is Mk-flat.*

Proof. The proof is immediate from Propositions 2.3 and 4.2. □

In module theory every projective module is flat. Now we see that this is true for certain special semimodules.

Proposition 4.4. *Let M be a cancellable left R-semimodule, where R is a cancellative completely subtractive semiring. Then every k-regular projective R-semimodule P is Mk-flat.*

Proof. By using [5, Theorem 19], P is isomorphic to a direct summand of a free semimodule F. By Corollary 4.3, F is Mk-flat. Hence, by using Proposition 2.3, P is Mk-flat. □

Corollary 4.5. *Let M be a k-regular left R-semimodule and R a cancellative completely subtractive semiring. Then every k-regular projective R-semimodule P is Mk-flat.*

Proof. We only need to show that M is cancellable. Since M is k-regular, then there exists a free R-semimodule F such that $\varphi : F \to M$ is surjective. Let $m_1 + m = m_2 + m$, where $m_1, m_2, m \in M$. Since φ is surjective, then $\varphi(a_1) + \varphi(a) = \varphi(a_2) + \varphi(a)$, where $\varphi(a_1) = m_1$, $\varphi(a) = m$, and $\varphi(a_2) = m_2$. Since φ is k-regular, then $a_1 + a + k_1 = a_2 + a + k_2$, where $k_1, k_2 \in \operatorname{Ker}\varphi$. Since F is cancellable, then $a_1 + k_1 = a_2 + k_2$. Hence $\varphi(a_1) = \varphi(a_2)$. □

Proposition 4.6. *Let M be a cancellable left R-semimodule. If V is a free R-semimodule, then the following assertions hold:*

(a) *V is Mk-flat;*

(b) *V^* is M-injective.*

Proof. By using Corollary 4.3, V is Mk-flat.

(i)⇒(ii). The proof is immediate from Proposition 3.3. □

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