IJMMS 2003:65, 4139–4144
PII. S0161171203209248
http://ijmms.hindawi.com

ON A CLASS OF HOLOMORPHIC FUNCTIONS DEFINED BY THE RUSCHEWEYH DERIVATIVE

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Received 23 September 2002

By using the Ruscheweyh operator $D^m f(z)$, $z \in U$, we will introduce a class of holomorphic functions, denoted by $M_n^m(\alpha)$, and obtain some inclusion relations.

2000 Mathematics Subject Classification: 30C45.

1. Introduction and preliminaries. Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C};\ |z| < 1\}. \tag{1.1}$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

We let

$$A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \cdots,\ z_1 \in U\} \tag{1.2}$$

with $A_1 = A$.

We let $\mathcal{H}[a,n]$ denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots, \quad z \in U. \tag{1.3}$$

If f and g are analytic in U, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $K = \{f \in A : \operatorname{Re}(zf''(z)/f'(z)) + 1 > 0,\ z \in U\}$ denote the class of normalized convex functions in U. We use the following subordination results.

Lemma 1.1 (Miller and Mocanu [2, page 71]). *Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z), \tag{1.4}$$

then $p(z) \prec g(z) \prec h(z)$*, where*

$$g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) \cdot t^{(\gamma/n)-1} dt. \tag{1.5}$$

The function g is convex and is the best (a,n) dominant.

Lemma 1.2 (Miller and Mocanu [1]). *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z), \tag{1.6}$$

where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_n z^n + \cdots$ is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \tag{1.7}$$

then

$$p(z) \prec g(z) \tag{1.8}$$

and this result is sharp.

Definition 1.3 [4]. For $f \in A$ and $m \geq 0$, the operator $D^m f$ is defined by

$$D^m f(z) = f(z) * \frac{z}{(1-z)^{m+1}} = \frac{z}{m!}\left[z^{m-1} f(z)\right]^{(m)}, \quad z \in U, \tag{1.9}$$

where $*$ stands for convolution.

Remark 1.4. We have

$$\begin{gathered} D^0 f(z) = f(z), \quad z \in U, \\ D^1 f(z) = z f'(z), \quad z \in U, \\ 2D^2 f(z) = z \cdot \left[D^1 f(z)\right]' + D^1 f(z), \\ (m+1) D^{m+1} f(z) = z\left[D^m f(z)\right]' + m D^m f(z). \end{gathered} \tag{1.10}$$

2. Main results

Definition 2.1. If $\alpha < 1$ and $m, n \in \mathbb{N}$, let $M_n^m(\alpha)$ denote the class of functions $f \in A_n$ which satisfy the inequality

$$\operatorname{Re}\left(D^m f\right)'(z) > \alpha. \tag{2.1}$$

Theorem 2.2. *If $\alpha < 1$ and $m, n \in \mathbb{N}$, then*

$$M_n^{m+1}(\alpha) \subset M_n^m(\delta), \tag{2.2}$$

where

$$\delta = \delta(\alpha, n, m) = 2\alpha - 1 + 2 \cdot (1-\alpha) \cdot \frac{m+1}{n}\beta\left(\frac{m+1}{n}\right),$$
$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt. \tag{2.3}$$

Proof. Let $f \in M_n^{m+1}(\alpha)$. By using the properties of the operator $D^m f(z)$, we have

$$(m+1)D^{m+1}f(z) = z \cdot (D^m f)'(z) + mD^m f(z), \quad z \in U. \tag{2.4}$$

Differentiating (2.4), we obtain

$$\begin{aligned}(m+1)\left[D^{m+1}f(z)\right]' &= z \cdot (D^m f)''(z) + (D^m f)'(z) + m(D^m f)'(z) \\ &= z(D^m f)''(z) + (m+1)(D^m f)'(z).\end{aligned} \tag{2.5}$$

If we let $p(z) = (D^m f)'(z)$, then $p'(z) = (D^m f)''(z)$ and (2.4) becomes

$$\left[D^{m+1}f(z)\right]' = p(z) + \frac{1}{m+1}z \cdot p'(z). \tag{2.6}$$

Since $f \in M_n^{m+1}(\alpha)$, by using Definition 2.1, we have

$$\operatorname{Re}\left[p(z) + \frac{1}{m+1}zp'(z)\right] > \alpha \tag{2.7}$$

which is equivalent to

$$p(z) + \frac{1}{m+1}zp'(z) \prec \frac{1+(2\alpha-1)z}{1+z} \equiv h(z). \tag{2.8}$$

By using Lemma 1.1, we have

$$p(z) \prec g(z) \prec h(z), \tag{2.9}$$

where

$$g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} \cdot t^{(m+1)/n-1}dt. \tag{2.10}$$

The function g is convex and is the best dominant.

From $p(z) \prec g(z)$, it results that

$$\operatorname{Re} p(z) > \delta = g(1) = \delta(\alpha, n, m), \tag{2.11}$$

where

$$\begin{aligned} g(1) &= \frac{m+1}{n}\int_0^1 t^{(m+1)/n-1}\cdot\frac{1+(2\alpha-1)t}{1+t}dt \\ &= 2\alpha-1+2\cdot\frac{m+1}{n}\cdot(1-\alpha)\beta\left(\frac{m+1}{n}\right), \end{aligned} \tag{2.12}$$

from which we deduce that $M_n^{m+1}(\alpha)\subset M_n^m(\delta)$. □

For $n=1$, this result was obtained in [3].

Theorem 2.3. *Let g be a convex function, $g(0)=1$, and let h be a function such that*

$$h(z)=g(z)+\frac{1}{m+1}zg'(z). \tag{2.13}$$

If $f\in A_n$ and verifies the differential subordination

$$(D^{m+1}f)'(z)\prec h(z), \tag{2.14}$$

then

$$(D^m f)'(z)\prec g(z). \tag{2.15}$$

Proof. From

$$(m+1)D^{m+1}f(z)=z\cdot(D^m f)'(z)+mD^m f(z), \tag{2.16}$$

we obtain

$$\begin{aligned} (m+1)[D^{m+1}f(z)]' &= (D^m f)'(z)+z(D^m f)''(z)+m(D^m f)'(z) \\ &= z(D^m f)''(z)+(m+1)(D^m f)'(z). \end{aligned} \tag{2.17}$$

If we let $p(z)=(D^m f)'(z)$, then we obtain

$$[D^{m+1}f(z)]'=p(z)+\frac{1}{m+1}zp'(z) \tag{2.18}$$

and (2.14) becomes

$$p(z)+\frac{1}{m+1}zp'(z)\prec g(z)+\frac{1}{m+1}zg'(z)\equiv h(z). \tag{2.19}$$

By using Lemma 1.2, we have

$$p(z)\prec g(z),\quad \text{i.e., } (D^m f)'(z)\prec g(z). \tag{2.20}$$

□

For $n=1$, this result was obtained in [3].

Theorem 2.4. *Let $h \in \mathcal{H}[U]$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\operatorname{Re}\left[1+\frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2(m+1)}, \quad m \geq 0. \tag{2.21}$$

If $f \in A_n$ and verifies the differential subordination

$$[D^{m+1}f(z)]' \prec h(z), \quad z \in U, \tag{2.22}$$

then

$$[D^m f(z)]' \prec g(z), \tag{2.23}$$

where

$$g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1}dt. \tag{2.24}$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1, 2] shows that the function g is convex. From

$$(m+1)D^{m+1}f(z) = z[D^m f(z)]' + mD^m f(z), \tag{2.25}$$

we obtain

$$(m+1)[D^{m+1}f(z)]' = z[D^m f(z)]'' + (m+1)[D^m f(z)]'. \tag{2.26}$$

If we let $p(z) = [D^m f(z)]'$, then we obtain

$$[D^{m+1}f(z)]' = p(z) + \frac{1}{m+1}zp'(z) \tag{2.27}$$

and (2.22) becomes

$$p(z) + \frac{1}{m+1}zp'(z) \prec h(z). \tag{2.28}$$

By using Lemma 1.1, we have

$$p(z) \prec g(z) = \frac{m+1}{nz^{(m+1)/n}} \int_0^z h(t)t^{(m+1)/n-1}dt. \tag{2.29}$$

□

Theorem 2.5. *Let g be a convex function, $g(0) = 1$, and*

$$h(z) = g(z) + nzg'(z). \tag{2.30}$$

If $f \in A_n$ and verifies the differential subordination

$$[D^m f(z)]' \prec h(z), \quad z \in U, \tag{2.31}$$

then

$$\frac{D^m f(z)}{z} \prec g(z). \tag{2.32}$$

Proof. We let $p(z) = D^m f(z)/z$, $z \in U$, and we obtain

$$D^m f(z) = zp(z). \tag{2.33}$$

By differentiating, we obtain

$$[D^m f(z)]' = p(z) + zp'(z), \quad z \in U. \tag{2.34}$$

Then (2.31) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z). \tag{2.35}$$

By using Lemma 1.2, we have (1.8). □

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