We are interested in the investigation of the orthogonality (in the sense of Birkhoff) of the range of an elementary operator and its kernel.

2000 Mathematics Subject Classification: 47B47, 47A30, 47B10.

1. Introduction. Let $H$ be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on $H$ into itself. Given $A, B \in B(H)$, we define the generalized derivation $\delta_{A,B} : B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - XB$ and the elementary operator derivation $\Delta_{A,B} : B(H) \to B(H)$ by $\Delta_{A,B}(X) = AXB - X$. Denote $\delta_{A,A} = \delta_A$, $\Delta_{A,A} = \Delta_A$.

In [1, Theorem 1.7], Anderson shows that if $A$ is normal and commutes with $T$, then, for all $X \in B(H)$,

$$\|T + \delta_A(X)\| \geq \|T\|. \quad (1.1)$$

It is shown in [9] that if the pair $(A, B)$ has the Fuglede-Putnam property (in particular, if $A$ and $B$ are normal operators) and $AT = TB$, then, for all $X \in B(H)$,

$$\|T + \delta_{A,B}(X)\| \geq \|T\|. \quad (1.2)$$

Duggal [3] showed that the above inequality (1.2) is also true when $\delta_{A,B}$ is replaced by $\Delta_{A,B}$. The related inequality (1.1) was obtained by the author [10], showing that if the pair $(A, B)$ has the Fuglede-Putnam property $(FP)_{C_p}$, then

$$\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p \quad (1.3)$$

for all $X \in B(H)$, where $C_p$ is the von Neumann-Schatten class, $1 \leq p < \infty$, and $\| \cdot \|_p$ is its norm for all $X \in B(H)$ and for all $T \in C_p \cap \ker \delta_{A,B}$. In all of the above results, $A$ was not arbitrary. In fact, certain normality-like assumptions have been imposed on $A$. A characterization of $T \in C_p$ for $1 < p < \infty$, which is orthogonal to $R(\delta_A|C_p)$ (the range of $\delta_A|C_p$) for a general operator $A$, has...
been carried out by Kittaneh [6], showing that if \( T \) has the polar decomposition \( T = U|T| \), then

\[
||T + \delta_A(X)||_p \geq ||T||_p
\]

for all \( X \in C_p \ (1 < p < \infty) \) if and only if \( |T|^{p-1}U^* \in \ker \delta_A \). By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if \( T \) has the polar decomposition \( T = U|T| \), then

\[
||T + \delta_{A,B}(X)||_p \geq ||T||_p
\]

for all \( X \in C_p \ (1 < p < \infty) \) if and only if \( |T|^{p-1}U^* \in \ker \delta_{B,A} \). By a simple modification in the proof of the above inequality, we can prove that this inequality is also true in the general case, that is, if \( T \) has the polar decomposition \( T = U|T| \), then

\[
||T + E_{A,B}(X)||_p \geq ||T||_p
\]

for all \( X \in C_p \ (1 < p < \infty) \) if and only if \( T \in \ker E_{A,B} \). In Sections 1, 2, 3, and 4, we prove these results in the case where we consider \( E_{A,B} \) instead of \( \delta_{A,B} \), which leads us to prove that if \( T \in C_p \) and \( \ker E_{A,B} \subseteq \ker E^{*}_{A,B} \), then

\[
||T + E_{A,B}(X)||_p \geq ||T||_p
\]

for all \( X \in C_p \ (1 < p < \infty) \) if and only if \( T \in \ker E_{A,B} \). In Sections 5, 6, and 7, we minimize the map \( ||S + E_{A,B}(X)||_p \) and we classify its critical points.

2. Preliminaries. Let \( T \in B(H) \) be compact and let \( s_1(X) \geq s_2(X) \geq \cdots \geq 0 \) denote the singular values of \( T \), that is, the eigenvalues of \( |T| = (T^*T)^{1/2} \) arranged in their decreasing order. The operator \( T \) is said to belong to the Schatten \( p \)-class \( C_p \) if

\[
||T||_p = \left[ \sum_{i=1}^{\infty} s_j(T)^p \right]^{1/p} = \left[ \text{tr}(T)^p \right]^{1/p}, \quad 1 \leq p < \infty,
\]

where \( \text{tr} \) denotes the trace functional. Hence, \( C_1 \) is the trace class, \( C_2 \) is the Hilbert-Schmidt class, and \( C_\infty \) is the class of compact operators with

\[
||T||_\infty = s_1(T) = \sup_{\|f\|=1} \|Tf\|
\]

denoting the usual operator norm. For the general theory of the Schatten \( p \)-classes, the reader is referred to [7, 11].

Recall that the norm \( \| \cdot \| \) of the \( B \)-space \( V \) is said to be Gateaux differentiable at nonzero elements \( x \in V \) if

\[
\lim_{t \to 0, t \in \mathbb{R}} \frac{\|x + ty\| - \|x\|}{t} = \mathcal{R}D_x(y)
\]

for all \( y \in V \). Here \( \mathbb{R} \) denotes the set of reals, \( \mathcal{R} \) denotes the real part, and \( D_x \) is the unique support functional (in the dual space \( V^* \)) such that \( \|D_x\| = 1 \) and \( D_x(x) = \|x\| \). The Gateaux differentiability of the norm at \( x \) implies that \( x \) is a smooth point of the sphere of radius \( \|x\| \).
It is well known (see [7] and the references therein) that, for \(1 < p < \infty\), \(C_p\) is a uniformly convex Banach space. Therefore, every nonzero \(T \in C_p\) is a smooth point and, in this case, the support functional of \(T\) is given by

\[
D_T(X) = \text{tr}\left( \frac{|T|^{p-1}UX^*}{\|T\|_p^{p-1}} \right)
\]

(2.4)

for all \(X \in C_p\), where \(T = U|T|\) is the polar decomposition of \(T\).

**Definition 2.1.** Let \(E\) be a complex Banach space. We define the orthogonality in \(E\). We say that \(b \in E\) is orthogonal to \(a \in E\) if, for all complex \(\lambda\), there holds

\[
\|a + \lambda b\| \geq \|a\|.
\]

(2.5)

This definition has a natural geometric interpretation, namely, \(b \perp a\) if and only if the complex line \(\{a + \lambda b \mid \lambda \in \mathbb{C}\}\) is disjoint with the open ball \(K(0, \|a\|)\), that is, if and only if this complex line is a tangent one. Note that if \(b\) is orthogonal to \(a\), then \(a\) needs not be orthogonal to \(b\). If \(E\) is a Hilbert space, then from (2.5), it follows that \(\langle a, b \rangle = 0\), that is, orthogonality in the usual sense.

### 3. The elementary operators AXB − CXD

**Lemma 3.1.** Let \(A, B \in B(H)\). The following statements are equivalent:

1. The pair \((A, B)\) has the property \((FP)_{C_p}\), \(1 \leq p < \infty\);
2. If \(AT = TB\), where \(T \in C_p\), then \(R(T)\) reduces \(A\), \(\ker(T) \perp\) reduces \(B\), and \(A|_{\overline{R(T)}}\) and \(B|_{\ker(T) \perp}\) are normal operators.

**Proof.** (1)⇒(2). Since \(C_p\) is a bilateral ideal and \(T \in C_p\), then \(AT \in C_p\). Hence as \(AT = TB\) and \((A, B)\) satisfies \((FP)_{C_p}\), \(A^*T = TB^*\), and so, \(R(T)\) and \(\ker(T) \perp\) are reducing subspaces for \(A\) and \(B\), respectively. Since \(A(AT) = (AT)B\) implies that \(A^*(AT) = (AT)B^*\) by \((FP)_{C_p}\) and the equality \(A^*T = TB^*\) implies that \(A^*A = AA^*T\), thus we see that \(A|_{\overline{R(T)}}\) is normal. Clearly, \((B^*, A^*)\) satisfies \((FP)_{C_p}\) and \(B^*T^* = T^*A^*\). Therefore, it follows from the above argument that \(B^*|_{\overline{R(T)}} = B|_{\ker(T) \perp}\) is normal.

(2)⇒(1). Let \(T \in C_p\) such that \(AT = TB\). Taking the two decompositions of \(H, H_1 = H = \overline{R(T)} \oplus \overline{R(T)^\perp}\) and \(H_2 = H = \ker(T) \perp \oplus \ker(T)\), then we can write \(A\) and \(B\) on \(H_1\) into \(H_2\), respectively:

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},
\]

(3.1)

where \(A_1\) and \(B_1\) are normal operators. Also we can write \(T\) and \(X\) on \(H_2\) into \(H_1\):

\[
T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.
\]

(3.2)
It follows from $AT = TB$ that $A_1T_1 = T_1B_1$. Since $A_1$ and $B_1$ are normal operators, then, by applying the Fuglede-Putnam theorem, we obtain $A_1^*T_1 = T_1B_1^*$, that is, $A^*T = TB$.

**Theorem 3.2.** Let $A, B \in B(H)$. If $A$ and $B$ are normal operators, then

$$||S - (AX - XB)||_p \geq ||S||_p$$  \hspace{1cm} (3.3)

for all $X \in C_p$ and for all $S \in \ker \delta_{A,B} \cap C_p$ $(1 \leq p < \infty)$.

**Proof.** Let $S = U|S|$ be the polar decomposition of $S$, where $U$ is an isometry such that $\ker U = \ker |S|$. Since

$$||U^*S||_p \leq ||U^*||_p ||S||_p = ||S||_p$$ \hspace{1cm} (3.4)

for all $S \in C_p$, then

$$||S - (AX - XB)||_p^p \geq ||U^*[S - (AX - XB)]||_p^p = ||S - U^*(AX - XB)||_p^p,$$ \hspace{1cm} (3.5)

and we have

$$|||S| - U^*(AX - XB)||_p^p \geq \sum_n |\langle [|S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p$$ \hspace{1cm} (3.6)

for any orthonormal basis $\{\varphi_n\}_{n \geq 1}$ of $H$. Since $AS = SB$, and $A$ and $B$ are normal operators, it follows from the Fuglede-Putnam theorem that $S^*A = BS^*$. Consequently, $S^*AS = BS^*S$ or $S^*SB = BS^*S$, that is, $B|S| = |S|B$. Since $|S|$ is a compact normal operator and commutes with $B$, there exists an orthonormal basis $\{f_k\} \cup \{g_m\}$ of $H$ such that $\{f_k\}$ consists of common eigenvectors of $B$ and $|S|$, and $\{g_m\}$ is an orthonormal basis of $\ker |S|$. Since $\{f_k\}$ is an orthonormal basis of the normal operator $B$, then there exists a scalar $\alpha_k$ such that $Bf_k = \alpha_kf_k$ and $B^*f_k = \overline{\alpha_k}f_k$. Consequently,

$$\langle U^*(AX - XB)f_k, |S|f_k \rangle = \langle S^*(AX - XB)f_k, f_k \rangle$$

$$= \langle (B(S^*X) - (S^*X)B)f_k, f_k \rangle = 0,$$ \hspace{1cm} (3.7)

that is, $\langle U^*(AX - XB)f_k, f_k \rangle = 0$.

In (3.6) take $\{\varphi_n\} = \{f_k\} \cup \{g_m\}$ as an orthonormal basis of $H$, then

$$\sum_n |\langle [|S| - U^*(AX - XB)]\varphi_n, \varphi_n \rangle|^p \geq \sum_k |\langle |S|f_k, f_k \rangle|^p + \sum_m |\langle U^*(AX - XB)g_m, g_m \rangle|^p$$ \hspace{1cm} (3.8)

$$\geq \sum_k |\langle |S|f_k, f_k \rangle|^p = ||S||_p^p.$$
Lemma 3.3. Let $A, B \in B(H)$ satisfying $(FP)_{C_p}$. Then

$$\|S + AX - XB\|_p \geq \|S\|_p$$

(3.9)

for every operator $S \in \ker \delta_{A,B} \cap C_p \ (1 < p < \infty)$ and for all $X \in C_p$.

Proof. If the pair $(A, B)$ satisfies the $(FP)_{C_p}$ property, then $R(S)$ reduces $A$, $\ker^\perp S$ reduces $B$, and $A|_{R(S)}$ and $B|_{\ker^\perp S}$ are normal operators. Letting $S_0 : \ker^\perp S \to R(S)$ be the quasiaffinity defined by setting $S_0 x = Sx$ for each $x \in \ker^\perp S$, it results that $\delta_{A_1, B_1} (S_0) = \delta_{A^*_1, B^*_1} (S_0) = 0$. Let $A = A_1 \oplus A_2$, with respect to $H = R(S) \oplus R(S)^\perp$, $A = B_1 \oplus B_2$, with respect to $H = \ker^\perp S \oplus \ker S$, and $X : R(S) \oplus R(S)^\perp \to \ker(S)^\perp \oplus \ker S$ have the matrix representation

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$  

(3.10)

Then we have

$$\|S - (AX - XB)\|_p = \left\| \begin{bmatrix} S_1 - (A_1 X_1 - X_1 B_1) \\ \ast \\ \ast \end{bmatrix} \right\|_p.$$  

(3.11)

The result of Gohberg and Krein [4] guarantees that

$$\|S - (AX - XB)\|_p \geq \|S_1 - (A_1 X_1 - X_1 B_1)\|_p.$$  

(3.12)

Since $A_1$ and $B_1$ are two normal operators, then it results from Theorem 3.5 that

$$\|S_1 - (A_1 X_1 - X_1 B_1)\|_p \geq \|S_1\|_p = \|S\|_p.$$  

(3.13)

Lemma 3.4 [6]. Let $u$ and $v$ be two elements of a Banach space $V$ with norm $\| \cdot \|$. If $u$ is a smooth point, then $D_u (v) = 0$ if and only if

$$\|u + zv\| \geq \|u\|$$

(3.14)

for all $z \in \mathbb{C}$ (the complex numbers).

Theorem 3.5. Let $A, B \in B(H)$ and $T \in C_p \ (1 < p < \infty)$. Then

$$\|T + \delta_{A,B} (X)\|_p \geq \|T\|_p$$

(3.15)

for all $X \in B(H)$ with $\Delta_{A,B} (X) \in C_p$ if and only if

$$\text{tr} (|T|^{p-1} U^* \delta_{A,B} (X)) = 0$$

(3.16)

for all such $X$. 


Proof. The theorem is an immediate consequence of equality (2.4) and Lemma 3.4.

Theorem 3.6. Let \( A, B \in B(H) \) and \( T \in \mathcal{C}_p \) \((1 < p < \infty)\). Then

\[
\|T + \delta_{A,B}(X)\|_p \geq \|T\|_p
\]

(3.17)

for all \( X \in \mathcal{C}_p \) if and only if \( \hat{T} = |T|^{p-1}U^* \in \ker \delta_{B,A} \).

Proof. By virtue of Theorem 3.5, it is sufficient to show that \( \text{tr}(\hat{T}\delta_{A,B}(X)) = 0 \) for all \( X \in \mathcal{C}_p \) if and only if \( \hat{T} \in \ker \delta_{B,A} \).

Choose \( X \) to be the rank-one operator \( f \otimes g \) for some arbitrary elements \( f \) and \( g \) in \( H \). Then \( \text{tr}(\hat{T}(AX - XB)) = \text{tr}(B\hat{T} - \hat{T}A)X = 0 \) implies that \( \langle \delta_{B,A}(\hat{T})f, g \rangle = 0 \) \( \Leftrightarrow \hat{T} \in \ker \delta_{B,A} \).

Conversely, assume that \( \hat{T} \in \ker \delta_{B,A} \), that is, \( B\hat{T} = \hat{T}A \). Since \( \hat{T}X \) and \( \hat{T}\delta_{B,A} \) are trace classes, then for all \( X \in \mathcal{C}_p \), we get

\[
\text{tr} \left( \hat{T}(AX - XB) \right) = \text{tr} \left( \hat{T}AX - \hat{T}XB \right) = \text{tr} \left( XB\hat{T} - X\hat{T}A \right) = \text{tr} \left( X\delta_{B,A}(\hat{T}) \right) = 0.
\]

(3.18)

Lemma 3.7. Let \( A, B \in B(H) \) and \( S \in \mathcal{C}_p \) such that \( \delta_{A,B}(T) = 0 = \delta_{A,B}^*(T) \). If \( A|S|^{p-1}U^* = |S|^{p-1}U^*B \), where \( p > 1 \) and \( S = U|S| \) is the polar decomposition of \( S \), then \( A|S|U^* = |S|U^*B \).

Proof. If \( T = |S|^{p-1} \), then

\[
ATU^* = TU^*B.
\]

(3.19)

We prove that

\[
AT^nU^* = T^nU^*B
\]

(3.20)

for all \( n \geq 1 \). If \( S = U|S| \), then

\[
\ker U = \ker |S| = \ker |S|^{p-1} = \ker T, \\
(ker U)^\perp = (ker T)^\perp = R(T).
\]

(3.21)

This shows that the projection \( U^*U \) onto \( (ker T)^\perp \) satisfies \( U^*UT = T \) and \( TU^*UT = T^2 \). By taking the adjoints of (3.19) and since \( A \) and \( B \) are normal operators applying Fuglede-Putnam theorem, we get \( BUT = UTA \) and \( AT^2 = ATU^*UT = TU^*BUT = TU^*UTA = T^2A \).

Since \( A \) commutes with the positive operator \( T^2 \), \( A \) commutes with its square roots, that is,

\[
AT = TA.
\]

(3.22)

By (3.19) and (3.22) we obtain (3.20). Let \( f(t) \) be the map defined on \( \sigma(T) \subset \mathbb{R}^+ \) by \( f(t) = t^{1/(p-1)} \) \((1 < p < \infty)\). Since \( f \) is the uniform limit of a sequence
of polynomials without constant term (since \( f(0) = 0 \)), it follows from (3.20) that \( AP_1(T)U^* = P_1(T)U^*B \). Therefore, \( AT^1/(p-1)U^* = U^*T^1/(p-1)B \). \( \square \)

**Theorem 3.8.** Let \( A \) and \( B \) be operators in \( B(H) \) such that \( \delta_{A,B}(T) = 0 = \delta_{A,B}^*(T) \). Then \( T \in \ker \Delta_{A,B} \cap C_p \) if and only if

\[
||S + \delta_{A,B}(X)||_p \geq ||S||_p \tag{3.23}
\]

for all \( X \in C_p \).

**Proof.** If \( S \in \ker \Delta_{A,B} \), then it follows from Lemma 3.3 that

\[
||S + \delta_{A,B}(X)||_p \geq ||S||_p \tag{3.24}
\]

for all \( X \in C_p \). Conversely, if

\[
||S + \delta_{A,B}(X)||_p \geq ||S||_p \tag{3.25}
\]

for all \( X \in C_p \), then, from Theorem 3.6,

\[
A|S|^{p-1}U^* = |S|^{p-1}U^*B. \tag{3.26}
\]

Since \( \delta_{A,B}(S) = 0 = \delta_{A,B}^*(S) \),

\[
A^*|S|^{p-1}U^* = |S|^{p-1}U^*B^*. \tag{3.27}
\]

By taking adjoints, we get

\[
AU|S|^{p-1} = U|S|^{p-1}B. \tag{3.28}
\]

From Lemma 3.7, it follows that \( AU|S| = U|S|B \), that is, \( S \in \ker \Delta_{A,B} \). \( \square \)

**Remark 3.9.** (1) It is well known that the Hilbert-Schmidt class \( C_2 \) is a Hilbert space under the inner product \( \langle Y, Z \rangle = \text{tr} Z^*Y \).

We remark here that for the Hilbert-Schmidt norm \( \| \cdot \|_2 \), the orthogonality result in Theorem 3.8 is to be understood in the usual Hilbert-space sense. Note in the case where \( I = C_2 \) that

\[
||T + \delta_{A,B}(X)||_2^2 = ||\delta_{A,B}(X)||_2^2 + ||T||_2^2 \tag{3.29}
\]

for all \( X \in C_2 \) if and only if \( AT^* = T^*B \). This can be seen as an immediate consequence of the fact that

\[
R(\delta_{A,B}(C_2)) = \ker (\delta_{A,B}(C_2))^* = \ker (\delta_{B^*,A^*}(C_2)). \tag{3.30}
\]

(2) It is known [2] that if \( A \) and \( B \) are contractions and \( S \in C_p \), then \( \delta_{A^*,B^*}(S) = \delta_{A,B}(S) = 0 \). Hence

\[
||S + \delta_{A,B}(X)||_p \geq ||S||_p \tag{3.31}
\]

holds for all \( X \in C_p \) if and only if \( S \in \ker (\delta_{A,B}(C_p)) \).
If $A = B$, then the following counterexample shows that Theorem 3.8 does not hold if $p < 1$. Take $p = 1/2$ and

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix},$$  \quad (3.32)

where $\alpha$ is real such that $0 < \alpha < 1$. We have

$$S - (AX - XA) = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$ \quad (3.33)

and, for eigenvectors $\beta_1 = 1 - \alpha$, $\beta_2 = 1 + \alpha$. Then

$$\|S - (AX - XA)\|_{1/2} = \left[ (1 - \alpha)^{1/2} + (1 + \alpha)^{1/2} \right]^2 < 4 = \|S\|_{1/2}. \quad (3.34)$$

**Corollary 3.10.** Let $A, B \in L(H)$. Then

$$\|S + AX - XB\|_p \geq \|S\|_p \quad (3.35)$$

if and only if $S \in \ker \delta_{A,B} \cap C_p$ and for all $X \in C_p$, in each of the following cases:

1. if $A, B \in L(H)$ such that $\|Ax\| \geq \|x\| \geq \|Bx\|$ for all $x \in H$,
2. if $A$ is invertible and $B$ is such that $\|A^{-1}\| \|B\| \leq 1$.

**Proof.** The result of Tong [13, Lemma 1] guarantees that the above condition implies that, for all $T \in \ker(\delta_{A,B}|_{K(H)})$, $R(T)$ reduces $A$, $\ker(T) \perp$ reduces $B$, and $A|_{R(T)}$ and $B|_{\ker(T)}$ are unitary operators. Hence it results from Lemma 3.1 that the pair $(A, B)$ has the property $(FP)_{K(H)}$ and the results hold by Theorem 3.8. Here $K(H)$ is the ideal of compact operators.

The above inequality holds in particular if $A = B$ is isometric; in other words, $\|Ax\| = \|x\|$ for all $x \in H$.

2. In this case, it suffices to take $A_1 = \|B\|^{-1}A$, $B_1 = \|B\|^{-1}B$. Then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$ and the result holds by (1) for all $x \in H$. \qed

**4. Orthogonality and the elementary operators** $AXB - CXD$. Let $H$ be a separable infinite-dimensional complex Hilbert space and let $B(H)$ denote the algebra of all bounded operators on $H$ into itself. Given $A, B, C, D$ normal operators in $B(H)$ such that $AC = CA$, $BD = DB$, we define the elementary operator $\Psi : B(H) \rightarrow B(H)$ by $\Psi(X) = AXB - CXD$. We prove that if $T \in C_p \quad (1 < p < \infty)$, then $\|T + \Phi(X)\|_p \geq \|T\|_p$ if and only if $T \in \ker \Phi$ for all $X \in C_p$.

By the same argument used in the proofs of Theorems 3.5 and 3.6, we prove the following theorems.

**Theorem 4.1.** Let $A, B, C, D \in B(H)$ and $T \in C_p \quad (1 < p < \infty)$. Then

$$\|T + \Psi(X)\|_p \geq \|T\|_p \quad (4.1)$$
for all $X \in B(H)$ with $\Psi(X) \in C_p$ if and only if
\[ \text{tr}(|T|^{p-1}U^*\Psi(X)) = 0 \] (4.2)
for all such $X$.

**Theorem 4.2.** Let $A, B, C, D \in B(H)$ and $T \in C_p\ (1 < p < \infty)$. Then
\[ ||T + \Psi(X)||_p \geq ||T||_p \] (4.3)
for all $X \in C_p$ if and only if $\tilde{T} = |T|^{p-1}U^* \in \text{ker}\Psi$.

**Lemma 4.3.** Let $A, B \in B(H)$ be normal operators and $AB = BA$. Suppose that $ASB = BSA$, $S \in C_p\ (1 < p < \infty)$. If
\[ AU|S|^{p-1}B = BU|S|^{p-1}A, \] (4.4)
then
\[ AU|S|B = BU|S|A. \] (4.5)

**Proof.** Assume that $B^{-1} \in B(H)$. Then, from $ASB = BSA$ and $AB = BA$, we get $AB^{-1}S = SB^{-1}A$. Hence, applying the above lemma to the operators $AB^{-1}$, $B^{-1}A$, and $S$, we get
\[ AB^{-1}U|S|^{p-1} = U|S|^{p-1}B^{-1}A, \] (4.6)
which implies that
\[ AB^{-1}U|S| = U|S|B^{-1}A. \] (4.7)

Multiply (4.6) and (4.7) at right and left by $B$ to obtain
\[ BAB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}AB \] (4.8)
or
\[ ABB^{-1}U|S|^{p-1}B = BU|S|^{p-1}B^{-1}BA, \] (4.9)
that is,
\[ AU|S|^{p-1}B = BU|S|^{p-1}A, \] (4.10)
which implies that
\[ AU|S|B = BU|S|A. \] (4.11)

Consider now the case when $B$ is injective, that is, $\text{ker}\ B = \{0\}$. Let
\[ \Delta_n = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{n} \right\} \] (4.12)
and let $E_B(\Delta_n)$ be the corresponding spectral projector.
Putting

\[ P_n = I - E_B(\Delta_n), \]  \hspace{1cm} (4.13)

the subspace \( P_nH \) reduces both operators \( A \) and \( B \) (since they commute and are normal). Hence, with respect to the decomposition

\[ H = (I - P_n)H \oplus P_nH, \]

\[ A = \begin{bmatrix} A_1(n) & 0 \\ 0 & A_2(n) \end{bmatrix}, \quad B = \begin{bmatrix} B_1(n) & 0 \\ 0 & B_2(n) \end{bmatrix}, \]

\[ S = \begin{bmatrix} S_{11}(n) & S_{12}(n) \\ S_{21}(n) & S_{22}(n) \end{bmatrix}, \quad X = \begin{bmatrix} X_{11}(n) & X_{12}(n) \\ X_{21}(n) & X_{22}(n) \end{bmatrix}, \]

it is easy to see that \( B_2^{(n)} \) acting on \( P_nH \) is invertible. Then, from \( ASB = BSA \), it follows that

\[ A_2^{(n)} S_{22}(n) B_2^{(n)} = B_2^{(n)} S_{22}(n) A_2^{(n)}, \]  \hspace{1cm} (4.15)

and, from \( AB = BA \), we get \( A_2 B_2 = B_2 A_2 \). Since

\[ AU|S|^p B = BU|S|^p A, \]  \hspace{1cm} (4.16)

according to the first part of the proof, it follows that

\[ A_2^{(n)} U|S_{22}(n)|^{p-1} B_2^{(n)} = B_2^{(n)} U|S_{22}(n)|^{p-1} A_2^{(n)}, \]  \hspace{1cm} (4.17)

which implies that

\[ A_2^{(n)} U|S_{22}(n)| B_2^{(n)} = B_2^{(n)} U|S_{22}(n)| A_2^{(n)}, \]  \hspace{1cm} (4.18)

so we have \( AU|S|B = BU|S|A \). Assume now \( \ker A \cap \ker B = \{0\} \).

Then \( \ker B \) reduces \( A \) and \( P_{\ker B} A P_{\ker B} \) is injective. Let \( H = \ker B \oplus H_1 \) (\( H_1 = H \ominus \ker B \)). Then we have

\[ A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \]

\[ \text{where } A_1, B_2 \text{ are injective and their ranges are dense in subspaces they act on. We have} \]

\[ ASB - BSA = \begin{bmatrix} 0 & A_1 S_{12} B_2 \\ -B_2 S_{21} A_1 & A_2 S_{22} B_2 - B_2 S_{22} A_2 \end{bmatrix}. \]  \hspace{1cm} (4.20)
Now, if $ASB = BSA$, then $A_2 S_{22} B_2 = B_2 S_{22} A_2$, $B_2 S_{21} A_1 = 0$, and $A_1 S_{12} B_2 = 0$, that is, $S_{21} = S_{12} = 0$. It follows that

\[ S = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}. \] (4.21)

Since $A_2 B_2 = B_2 A_2$, $A_2 S_{22} B_2 = B_2 S_{22} A_2$, and $B_2$ is injective, and we have already proved that

\[ A_2 U |S_{22}| B_2^{-1} = B_2 U |S_{22}| A_2^{-1}, \] (4.22)

implies

\[ A_2 U |S_{22}| B_2 = B_2 U |S_{22}| A_2, \] (4.23)

so we have $AU|S|B = BU|S|A$.

Let $\Phi(X) = AXB - BXA$. We prove the following theorem.

**Theorem 4.4.** Let $A, B \in B(H)$ be normal operators, $AB = BA$, and $S \in C_p$ $(1 < p < \infty)$. Then $S \in \ker \Phi$ if and only if

\[ ||S - (AXB - BXA)||_p \geq ||S||_p \] (4.24)

for all $X \in C_p$.

**Proof.** If $S \in \ker \Phi$, then, from [13, Theorem 3.4], it follows that

\[ ||S + \Phi(X)||_p \geq ||S||_p \] (4.25)

for all $X \in C_p$. Conversely, if

\[ ||S + \Phi(X)||_p \geq ||S||_p \] (4.26)

for all $X \in C_p$, then, from **Theorem 4.2**, \n
\[ A |S|^{p-1} U^* B = B |S|^{p-1} U^* A. \] (4.27)

Since $A$ and $B$ are normal operators applying Fuglede-Putnam theorem, we get $A^* |S|^{p-1} U^* B^* = B^* |S|^{p-1} U^* A^*$. By taking adjoints, we get $AU|S|^{p-1} B = BU|S|^{p-1} A$.

From **Lemma 4.3**, it follows that $AU|S|B = BU|S|A$, that is, $S \in \ker \Phi$.

Let $\Psi(X) = AXB - CXD$.

**Theorem 4.5.** Let $A, B, C, D \in B(H)$ be normal operators, $AC = CA$, $BD = DB$, and $S \in C_p$ $(1 < p < \infty)$. Then $S \in \ker \Psi$ if and only if

\[ ||S - (AXB - CXD)||_p \geq ||S||_p \] (4.28)

for all $X \in C_p$. 
**Proof.** It suffices to take the Hilbert space $H \oplus H$ and the operators

\[
A^- = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad B^- = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix},
\]
\[S^- = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad X^- = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}
\]

and apply Theorem 4.4.

**Remark 4.6.** The results of the above theorems can be obtained when the normality of $A$ and $B$ is replaced by some other condition, in particular, if $|A| = |B|$, $|A^*| = |B^*|$. In this case, it suffices to take

\[
A^- = \begin{bmatrix} 0 & A^* \\ B & 0 \end{bmatrix}, \quad B^- = \begin{bmatrix} 0 & B^* \\ A & 0 \end{bmatrix},
\]
\[S^- = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \quad X^- = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}
\]

and apply Lemma 4.3 and Theorem 4.4.

5. On minimizing $\|AX - XB - T\|_p^p$. Maher [8, Theorem 3.2] shows that if $A$ is normal and $S \in \ker \delta_A \cap C_p$ $(1 \leq p < \infty)$, then the map $F_p$ defined by $F_p(X) = \|S - (AX - XA)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AV - VA = 0$.

In this section, we prove that if the pair $(A, B)$ has the property $(FP)_{C_p}$ (i.e., $AT = TB$, where $T \in C_p$, implies $A^*T = TB^*$), $1 \leq p < \infty$, and $S \in \ker \delta_{A,B} \cap C_p$, then the map $F_p$ defined by $F_p(X) = \|S - (AX - XB)\|_p^p$ has a global minimizer at $V$ if, and for $1 < p < \infty$ only if, $AV - VB = 0$. In other words, we have

\[
\|S - (AX - XB)\|_p^p \geq \|T\|_p^p
\]

if, and for $1 < p < \infty$ only if, $AV - VB = 0$. Thus in Halmos’ terminology [5], the zero commutator is the commutator approximant in $C_p$ of $T$. Additionally, we show that if the pair $(A, B)$ has the property $(FP)_{C_p}$ and $S \in \ker \delta_{A,B} \cap C_p$ $(1 < p < \infty)$, then the map $F_p$ has a critical point at $W$ if and only if $AW - WB = 0$, that is, if $\mathcal{D}_{W} F_p$ is the Frechet derivative at $W$ of $F_p$, the set $\{W \in B(H) : \mathcal{D}_{W} F_p = 0\}$ coincides with $\ker \delta_{A,B}$ (the kernel of $\delta_{A,B}$).

**Theorem 5.1** [9]. If $1 < p < \infty$, then the map $F_p : C_p \to \mathbb{R}^+$ defined by $X \mapsto \|X\|_p^p$ is differentiable at every $X \in C_p$ with derivative $\partial_X F_p$ given by $\partial_X F_p(T) = p \text{Re} \text{Tr}(X|T|^{-1}U^*T)$, where $\text{tr}$ denotes trace, $\text{Re} z$ is the real part of a complex number $z$, and $X = U|X|$ is the polar decomposition of $X$. If $\dim H < \infty$, then the same result holds for $0 < p \leq 1$ at every invertible $X$.

**Theorem 5.2** [9]. If $\mathcal{U}$ is a convex set of $C_p$, $1 < p < \infty$, then the map $X \mapsto \|X\|_p^p$, where $X \in \mathcal{U}$, has at most a global minimizer.
**Definition 5.3.** Let \( \mathcal{U}(A, B) = \{X \in B(H) : AX - XB \in C_p\} \) and let \( F_p : \mathcal{U} \rightarrow \mathbb{R}^+ \) be the map defined by \( F_p(X) = \|T - (AX - XB)\|_p^p \), where \( T \in \ker \delta_{A,B} \cap C_p, 1 \leq p < \infty \).

**6. Main results.** By simple modifications in the proof of Lemma 3.7, we can prove the following lemma.

**Lemma 6.1.** Let \( A, B \in B(H) \) and \( C \in B(H) \) such that the pair \( (A, B) \) has the property \( (FP)_{B(H)} \). If \( A|S|^{p^{-1}}U^* = |S|^{p^{-1}}U^*B \), where \( p > 1 \) and \( S = U|S| \) is the polar decomposition of \( S \), then \( A|S|U^* = |S|U^*B \).

**Theorem 6.2.** Let \( A, B \in \mathcal{L}(H) \). If the pair \( (A, B) \) has the property \( (FP)_{C_p} \) and \( S \in C_p \) such that \( AS = SB \), then

1. for \( 1 \leq p < \infty \), the map \( F_p \) has a global minimizer at \( W \) if, and for \( 1 < p < \infty \) only if, \( AW - WB = 0 \);
2. for \( 1 < p < \infty \), the map \( F_p \) has a critical point at \( W \) if and only if \( AW - WB = 0 \);
3. for \( 0 < p \leq 1 \) \( \dim \mathcal{U} < \infty \) and \( S - (AW - WB) \) is invertible, then \( F_p \) has a critical point at \( W \) if \( AW - WB = 0 \).

**Proof.** Since the pair \( (A, B) \) has the property \( (FP)_{C_p} \), it follows from Lemma 3.3 that

\[ \|S - (AX - XB)\|_p^p \geq \|S\|_p^p, \tag{6.1} \]

that is, \( F_p(X) \geq F_p(W) \).

Conversely, if \( F_p \) has a minimum, then

\[ \|S - (AW - WB)\|_p^p = \|S\|_p^p. \tag{6.2} \]

Since \( \mathcal{U} \) is convex, the set \( \mathcal{V} = \{S - (AX - XB) ; X \in \mathcal{U}\} \) is also convex. Thus Theorem 5.2 implies that

\[ S - (AW - WB) = S. \tag{6.3} \]

(2) Let \( W, S \in \mathcal{U} \) and let \( \phi \) and \( \varphi \) be two maps defined, respectively, by \( \phi : X \rightarrow S - (AX - XB) \) and \( \varphi : X \rightarrow \|X\|_p^p \).

Since the Frechet derivative of \( F_p \) is given by

\[ \mathcal{D}_WF_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h}, \tag{6.4} \]

it follows that

\[ \mathcal{D}_WF_p(T) = \left[\mathcal{D}_{S - (AW - WB)}\right](TB - AT). \tag{6.5} \]
If \( W \) is a critical point of \( F_p \), then \( \partial W F_p(T) = 0 \) for all \( T \in \mathcal{U} \). By applying Theorem 5.1, we get

\[
\partial W F_p(T) = p \text{Re} \text{tr} \left[ |S - (AW - WB)|^{p-1} W^*(TB - AT) \right] = p \text{Re} \text{tr} \left[ Y(TB - AT) \right] = 0,
\]

(6.6)

where \( S - (AW - WB) = W|S - (AW - WB)| \) is the polar decomposition of the operator \( S - (AW - WB) \) and \( Y = |S - (AW - WB)|^{p-1} W^* \). An easy calculation shows that \( BY - YA = 0 \), that is,

\[
A|S - (AW - WB)|^{p-1} W^* = |S - (AW - WB)|^{p-1} W^* B.
\]

(6.7)

It follows from Lemma 6.1 that

\[
A|S - (AW - WB)| W^* = |S - (AW - WB)| W^* B.
\]

(6.8)

By taking adjoints and since the pair \((A,B)\) has the property \((FP)_{cp}\), we get \( A(T - (AW - WB)) = (T - (AW - WB))B \). Then \( A(AW - WB) = (AW - WB)B \). Hence

\[
AW - WB \in R(\delta_{A,B}) \cap \ker \delta_{A,B}.
\]

(6.9)

By the same argument used in the proof of Lemma 6.1 we can prove that

\[
||S - (AX - XB)|| \geq ||S||
\]

(6.10)

for all \( X \in B(H) \) and for all \( T \in B(H) \) and it results that \( AW - WB = 0 \).

Conversely, if \( AW = WB \), then \( W \) is a minimum, and since \( F_p \) is differentiable, then \( W \) is a critical point.

(3) Suppose that \( \dim H < \infty \). If \( AW - WB = 0 \), then \( S \) is invertible by hypothesis. Also \( |S| \) is invertible, hence \( |S|^p \) exists for \( 0 < p \leq 1 \) taking \( Y = |S|^{p-1} U^* \), where \( S = U|S| \) is the polar decomposition of \( S \). Since \( AS = SB \) implies that \( S^* A = BS^* \), then \( S^* A S = BS^* S \), and this implies that \( |S|^2 B = B|S|^2 \) and \( |S|B = B|S| \).

Since \( S^* A = BS^* \), that is, \( |S|U^* A = B|S|U^* \), then \( |S|(U^* A - BU^*) = 0 \), and since \( B|S|^p = |S|^{p-1} B \), then

\[
BY - YA = B|S|^{p-1} U^* - |S|^{p-1} U^* A = |S|^{p-1} (BU^* - U^* A)
\]

(6.11)

so that \( BY - YA = 0 \) and \( \text{tr}[BY - YA] = 0 \) for every \( T \in B(H) \). Since \( S = S - (AW - WB) \), then

\[
0 = \text{tr}[YT - YAT] = \text{tr}[Y(TB - AT)]
= p \text{Re} \text{tr} \left[ Y(TB - AT) \right] = p \text{Re} \text{tr} \left[ |S|^{p-1} U^*(TB - AT) \right]
= (\partial T \phi)(TB - AT) = (\partial W F_p)(T).
\]

\( \Box \)
Remark 6.3. In Theorem 6.2, the implication "W is a critical point implies \( AW - WB = 0 \)" does not hold in the case \( 0 < p \leq 1 \) because the functional calculus argument involving the function \( t \mapsto t^{1/(p-1)} \), where \( 0 \leq t < \infty \), is only valid for \( 1 < p < \infty \).

7. On minimizing \( \| T - (AXB - CXD) \|_p^p \). In this section, we consider the elementary operator \( \Phi(X) = AXB - CXD \) and prove that if \( AC = CA, BD = DB \), and \( ASB = CSD, S \in C_p \), then, for \( 1 < p < \infty \), the map \( F_p \) defined by \( F_p(X) = \| T - (AXB - CXD) \|_p^p \) has a global minimizer at \( V \) if, and for \( 1 < p < \infty \) only if, \( AVB - CVD = 0 \). In other words, we have \( \| T - (AXB - CXD) \|_p^p \geq \| T \|_p^p \) if, and for \( 1 < p < \infty \) only if, \( AVB - CVD = 0 \). Additionally, we show that if \( AC = CA, BD = DB, \) and \( T \in \ker \Delta_{A,B} \cap C_p, 1 < p < \infty \), then the map \( F_p \) has a critical point at \( W \) if and only if \( AWB - CWD = 0 \), that is, if \( \mathcal{D}_W F_p \) is the Frechet derivative at \( W \) of \( F_p \), the set \( \{ W \in B(H) : \mathcal{D}_W F_p = 0 \} \) coincides with \( \ker \Phi \) (the kernel of \( \Phi \)).

Definition 7.1. Let \( \mathcal{U}(A,B) = \{ X \in B(H) : AXB - CXD \in C_p \} \) and let \( F_p : \mathcal{U} \to \mathbb{R}^+ \) be the map defined by \( F_p(X) = \| T - (AXB - CXD) \|_p^p \), where \( T \in \ker \Delta_{A,B} \cap C_p, 1 \leq p < \infty \).

The proof of the following lemma is similar to the proof of Lemma 4.3.

Lemma 7.2. Let \( A,B \in B(H) \) be normal commuting operators. Suppose that \( ASB = BSA, S \in C_p (1 < p < \infty) \). If
\[
A|S|^{p-1}U^*B = B|S|^{p-1}U^*A, \tag{7.1}
\]
then
\[
A|S|U^*B = B|S|U^*A. \tag{7.2}
\]

Theorem 7.3. Let \( A,B,C,D \in B(H) \) be normal operators such that \( AC = CA \) and \( BD = DB \). Assume that \( ASB = CSD, S \in C_p (1 < p < \infty) \). If \( A|S|^{p-1}U^*B = C|S|^{p-1}U^*D, \) then \( A|S|U^*B = C|S|U^*D \).

Proof. It suffices to take the Hilbert space \( H \oplus H \) and the operators
\[
A^- = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \quad B^- = \begin{bmatrix} C & 0 \\ 0 & B \end{bmatrix}, \quad S^- = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \tag{7.3}
\]
and apply Lemma 7.2. \( \Box \)

Theorem 7.4. Let \( A,B,C,D \in B(H) \) be normal operators, \( AC = CA \), and \( BD = DB \). Suppose that \( ASB = CSD, S \in C_p \). Then, for \( 1 \leq p < \infty \), the map \( F_p \) has a global minimizer at \( W \) if, and for \( 1 < p < \infty \) only if, \( AWB - CWD = 0 \).
**Proof.** If $AC = CA$, $BD = DB$, and $ASB = CSD$, $S \in C_p$, then, for $1 < p < \infty$, the result of Turnšek [14, Theorem 3.4] guarantees that

$$\|T - (AXB - CXD)\|_p^p \geq \|T\|_p^p,$$

(7.4)

that is, $F_p(X) \geq F_p(W)$. Conversely, if $F_p$ has a minimum, then

$$\|T - (AWB - CWD)\|_p^p = \|S\|_p^p.$$

(7.5)

Since $\mathcal{U}$ is convex, then the set $\mathcal{V} = \{T - (AXB - CXD) ; X \in \mathcal{U}\}$ is also convex. Thus Theorem 5.2 implies that $S - (AWB - CWD) = S$. \(\square\)

**Theorem 7.5.** Let $A$, $B$, $C$, and $D$ be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker \Phi \cap C_p$, then, for $1 < p < \infty$, the map $F_p$ has a critical point at $W$ if and only if $AWB - CWD = 0$.

**Proof.** Let $W, S \in \mathcal{U}$ and let $\phi$ and $\psi$ be two maps defined, respectively, by $\phi : X \mapsto S - (AXB - CXD)$ and $\psi : X \mapsto \|X\|_p^p$. Since the Frechet derivative of $F_p$ is given by

$$\mathcal{D}_W F_p(T) = \lim_{h \to 0} \frac{F_p(W + hT) - F_p(W)}{h},$$

(7.6)

it follows that $\mathcal{D}_W F_p(T) = [\mathcal{D}_{S - (AWB - CWD)}](BTA - DTC)$. If $W$ is a critical point of $F_p$, then $\mathcal{D}_W F_p(T) = 0$ for all $T \in \mathcal{U}$. By applying Theorem 5.1, we get

$$\mathcal{D}_W F_p(T) = p \text{ Retr} \left[ \|S - (AWB - CWD)\|_p^{p-1} W^* (BTA - DTC) \right]$$

(7.7)

where $S - (AWB - CWD) = W|S - (AWB - CWD)|$ is the polar decomposition of the operator $S - (AWB - CWD)$ and $Y = |S - (AWB - CWD)|^{p-1} W^*$. An easy calculation shows that $BYA - DYC = 0$, that is,

$$A|S - (AWB - CWD)|^{p-1} W^* B = C|S - (AWB - CWD)|^{p-1} W^* D.$$

(7.8)

It follows from Theorem 7.3 that

$$A|S - (AWB - CWD)| W^* B = C|S - (AWB - CWD)| W^* D.$$

(7.9)

By taking adjoints and since $A$ and $B$ are normal operators, applying Fuglede-Putnam theorem, we get $A(T - (AWB - CWD))B = C(T - (AWB - CWD))D$. Then $A(AW - WB)B = C(AWB - CWD)D$. Hence $AWB - CWD \in R(\Phi) \cap \ker \Phi$. By the same argument used in the proof of [13, Theorem 3.4], we can prove that

$$\|T - (AXB - CXD)\| \geq \|T\|$$

(7.10)

for all $T \in B(H)$. Hence $AWB - CWD = 0$. 
Conversely, if $AWB = CWD$, then $W$ is a minimum, and since $F_p$ is differentiable, then $W$ is a critical point.

**Theorem 7.6.** Let $A, B, C$, and $D$ be normal operators in $B(H)$ such that $AC = CA$ and $BD = DB$. If $S \in \ker \Phi \cap C_p$, $0 < p \leq 1$, $\dim H < \infty$, and $S - (AWB - CWD)$ is invertible, then $F_p$ has a critical point at $W$ if $AWB - CWD = 0$.

**Proof.** Suppose that $\dim H < \infty$. If $AWB - CWD = 0$, then $S$ is invertible by hypothesis. Also $|S|$ is invertible, hence $|S|^{p-1}$ exists for $0 < p \leq 1$. Taking $Y = |S|^{p-1}U^*$, where $S = U|S|$ is the polar decomposition of $S$, choose $X$ to be the rank-one operator $f \otimes g$ for some arbitrary elements $f$ and $g$ in $H \oplus H$. Then $\text{tr}(Y(AXB - CXD)) = \text{tr}(AYB - CYD)X = 0$ implies that $\langle \Psi(Y)f, g \rangle = 0 \iff Y \in \ker \Phi$, that is, $AYB - CYD = 0$ and $\text{tr}[(DYC - AYB)T] = 0$ for every $T \in B(H)$. Since $S = S - (AWB - CWD)$, then

$$0 = \text{tr}[YDTC - YATB] = \text{tr}[Y(DTC - ATB)]$$
$$= p \text{Re} \text{tr}[Y(DTC - ATB)] = p \text{Re} \text{tr}[|S|^{p-1}U^*(DTC - ATB)]$$
$$= (\mathbb{D}_T \Phi)(DTC - ATB) = (\mathbb{D}_W F_p)(T). \tag{7.11}$$

**Remark 7.7.** The set $\mathcal{F} = \{X : AXB - CXD \in C_p\}$ contains $C_p$; if $X \in C_p$, then $X \in \mathcal{F}$ and, for example, $I \in \mathcal{F}$ but $I \notin C_p$. If $A \in C_p$, the conclusions of Theorems 7.3, 7.4, 7.5, and 7.6 hold for all $X \in B(H)$.

For $n > 2$ the generalization of the above results to the elementary operators $\sum_{i=1}^n A_i X B_i$ is not possible. In [12], Shul’man stated that there exists a normally represented elementary operator of the form $\sum_{i=1}^n A_i X B_i$ with $n > 2$ such that $\text{asc} E > 1$, that is, the range and kernel have no trivial intersection.

**Acknowledgment.** This work was supported by the Research Center Project no. Math/1422/10.

**References**


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