IJMMS 2003:38, 2415–2423
PII. S0161171203209212
http://ijmms.hindawi.com
© Hindawi Publishing Corp.

ON m-ACCRETIVE SCHRÖDINGER-TYPE OPERATORS WITH SINGULAR POTENTIALS ON MANIFOLDS OF BOUNDED GEOMETRY

OGNJEN MILATOVIC

Received 20 September 2002

We consider a Schrödinger-type differential expression $\nabla^*\nabla + V$, where ∇ is a C^∞-bounded Hermitian connection on a Hermitian vector bundle E of bounded geometry over a manifold of bounded geometry (M,g) with positive C^∞-bounded measure $d\mu$, and V is a locally integrable linear bundle endomorphism. We define a realization of $\nabla^*\nabla + V$ in $L^2(E)$ and give a sufficient condition for its m-accretiveness. The proof essentially follows the scheme of T. Kato, but it requires the use of a more general version of Kato's inequality for Bochner Laplacian operator as well as a result on the positivity of solution to a certain differential equation on M.

2000 Mathematics Subject Classification: 35P05, 58J50, 47B25, 81Q10.

1. Introduction and the main result

1.1. The setting. Let (M,g) be a C^∞ Riemannian manifold without boundary, with metric g, $\dim M = n$. We will assume that M is connected. We will also assume that M has bounded geometry. Moreover, we will assume that we are given a positive C^∞-bounded measure $d\mu$, that is, in any local coordinates $x^1, x^2, \dots, x^n$, there exists a strictly positive C^∞-bounded density $\rho(x)$ such that $d\mu = \rho(x)dx^1 dx^2 \cdots dx^n$.

Let E be a Hermitian vector bundle over M. We will assume that E is a bundle of bounded geometry (i.e., it is supplied by an additional structure: trivializations of E on every canonical coordinate neighborhood U such that the corresponding matrix transition functions $h_{U,U'}$ on all intersections $U \bigcap U'$ of such neighborhoods are C^∞-bounded, that is, all derivatives $\partial_y^\alpha h_{U,U'}(y)$, where α is a multiindex, with respect to canonical coordinates, are bounded with bounds C_α which do not depend on the chosen pair U, U').

We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u,v) = \int_M \langle u(x), v(x)\rangle_{E_x} d\mu(x). \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle_{E_x}$ denotes the fiberwise inner product.

Let

$$\nabla : C^\infty(E) \longrightarrow C^\infty(T^*M \otimes E) \tag{1.2}$$

be a Hermitian connection on E which is C^∞-bounded as a linear differential operator, that is, in any canonical coordinate system U (with the chosen trivializations of $E|_U$ and $(T^*M \otimes E)|_U$), ∇ is written in the form

$$\nabla = \sum_{|\alpha| \le 1} a_\alpha(y)\partial_y^\alpha, \tag{1.3}$$

where α is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial_y^\beta a_\alpha(y)$ for any multiindex β are bounded by a constant C_β which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

$$H_V = \nabla^*\nabla + V. \tag{1.4}$$

Here

$$\nabla^* : C^\infty(T^*M \otimes E) \longrightarrow C^\infty(E) \tag{1.5}$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1), and V is a linear bundle endomorphism of E, that is, for every $x \in M$,

$$V(x) : E_x \longrightarrow E_x \tag{1.6}$$

is a linear operator.

We make the following assumption on V.

Assumption 1.1. Assume that $V \in L^p_{\text{loc}}(\operatorname{End} E)$, where
- (i) $p = 2n/(n+2)$ for $n \ge 3$,
- (ii) $p > 1$ for $n = 2$,
- (iii) $p = 1$ for $n = 1$.

We will use the following notations:

$$V_1(x) := \frac{V(x) + (V(x))^*}{2}, \quad V_2(x) := \frac{V(x) - (V(x))^*}{2i}, \quad x \in M, \tag{1.7}$$

where $i = \sqrt{-1}$ and $(V(x))^*$ denotes the adjoint of the linear operator (1.6) (in the sense of linear algebra).

By (1.7), for all $x \in M$, we have the following decomposition:

$$V(x) = V_1(x) + iV_2(x). \tag{1.8}$$

1.2. Sobolev space $W^{1,2}(E)$. By $W^{1,2}(E)$ we will denote the set of all $u \in L^2(E)$ such that $\nabla u \in L^2(T^*M \otimes E)$. It is well known (see, e.g., [5, Section A1.1]) that $W^{1,2}(E)$ is the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u,v)_1 := (u,v) + (\nabla u, \nabla v), \quad u,v \in C_c^\infty(E). \tag{1.9}$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

Since (M,g) and E have bounded geometry, by [5, Section A1.1], it follows that the usual Sobolev embedding theorem (see, e.g., [1, Theorem 2.21]) holds for $W^{1,2}(E)$.

1.3. A realization of H_V in $L^2(E)$. Let V be as in Assumption 1.1. We define an operator S associated to H_V as an operator in $L^2(E)$ given by $Su = H_V u$ with domain

$$\operatorname{Dom}(S) = \{u \in W^{1,2}(E) : H_V u \in L^2(E)\}. \tag{1.10}$$

Remark 1.2. We will show that for all $u \in W^{1,2}(E)$, we have $Vu \in L^1_{\text{loc}}(E)$ so that $H_V u$ in (1.10) can be understood in distributional sense.

Let $u \in W^{1,2}(E)$. For $n \geq 3$, by Section 1.2 above and the first part of Theorem 2.21 from Aubin [1], we have the following continuous embedding

$$W^{1,2}(E) \subset L^{p'}(E), \tag{1.11}$$

where $1/p' = 1/2 - 1/n$.

Let $p = 2n/(n+2)$ be as in Assumption 1.1. Since $1/p + 1/p' = 1$, by Hölder's inequality, it follows that $Vu \in L^1_{\text{loc}}(E)$.

For $n = 2$, by the first part of Theorem 2.21 from Aubin [1], we get the continuous embedding (1.11) for all $2 < p' < \infty$. By Assumption 1.1, for $n = 2$, we have $p > 1$. We may assume that $1 < p < 2$ (if $V \in L^t_{\text{loc}}(\operatorname{End} E)$ with $t \geq 2$, then $V \in L^p_{\text{loc}}(\operatorname{End} E)$ for all $1 < p < 2$). Given $1 < p < 2$, we can take $p' > 2$ such that $1/p + 1/p' = 1$. By Hölder's inequality, we have $Vu \in L^1_{\text{loc}}(E)$.

For $n = 1$, it is well known (see, e.g., the second part of Theorem 2.21 in [1]) that (1.11) holds with $p' = \infty$. By Assumption 1.1, for $n = 1$, we have $p = 1$. Thus, by Hölder's inequality, we have $Vu \in L^1_{\text{loc}}(E)$.

We now state the main result.

Theorem 1.3. *Assume that (M,g) is a manifold of bounded geometry with a positive C^∞-bounded measure $d\mu$. Assume that E is a Hermitian vector bundle of bounded geometry over M. Assume ∇ to be a C^∞-bounded Hermitian connection* *on E. Let V be as in Assumption 1.1. Moreover, assume that for all $x \in M$,*

$$V_1(x) \geq 0, \quad \text{as an operator } E_x \longrightarrow E_x, \tag{1.12}$$

where $V_1(x)$ is as in (1.7).

Then S is m-accretive.

Remark 1.4. The main source of inspiration for Theorem 1.3 was a result of Kato [3, Theorem I] which was proven for the operator $-\Delta + V$ on an open set $\Omega \subset \mathbb{R}^n$, where $-\Delta$ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V \in L^p_{\text{loc}}(\Omega)$, with p as in Assumption 1.1, is a complex-valued function such that $\operatorname{Re} V \geq 0$.

Let $d : C^\infty(M) \to \Omega^1(M)$ be the standard differential. Then $d^*d : C^\infty(M) \to C^\infty(M)$ is called the scalar Laplacian and will be denoted by Δ_M.

2. Proof of Theorem 1.3.

We will adopt the proof of [3, Theorem I] in our context. Throughout this section, we assume that all hypotheses of Theorem 1.3 are satisfied. We begin by introducing another realization of H_V.

2.1. Maximal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$.

We define an operator T associated to H_V as an operator $W^{1,2}(E) \to W^{-1,2}(E)$ given by $Tu = H_V u$ with domain

$$\operatorname{Dom}(T) = \{u \in W^{1,2}(E) : H_V u \in W^{-1,2}(E)\}. \tag{2.1}$$

Remark 2.1. Condition $H_V u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$ makes sense since $H_V u$ is a distributional section of E by Remark 1.2. Since $\nabla^*\nabla u \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$, it follows that the condition $H_V u \in W^{-1,2}(E)$ in (2.1) is equivalent to $Vu \in W^{-1,2}(E)$ for $u \in W^{1,2}(E)$.

Lemma 2.2. *The following inclusion holds: $C_c^\infty(E) \subset \operatorname{Dom}(T)$.*

Proof. Let $u \in C_c^\infty(E)$. Then $Vu \in L^p(E)$, where p is as in Assumption 1.1. By Remark 1.2, it follows that $W^{1,2}(E) \subset L^{p'}(E)$, where $1/p + 1/p' = 1$. By duality, we have $L^p(E) \subset W^{-1,2}(E)$. Thus $Vu \in W^{-1,2}(E)$, and hence $u \in \operatorname{Dom}(T)$. □

2.2. Minimal realization of H_V between $W^{1,2}(E)$ and $W^{-1,2}(E)$.

By T_0 we will denote the restriction of T with $\operatorname{Dom}(T_0) = C_c^\infty(E)$. Clearly, T_0 is a densely defined operator.

Remark 2.3. Since $\operatorname{Dom}(S)$, where S is as in (1.10), does not necessarily contain $C_c^\infty(E)$, there is no minimal realization of H_V in $L^2(E)$ (in the sense of Section 2.2).

2.3. Maximal and minimal realization of H_{V^*}.

In what follows, we will denote by T' and T_0' the maximal and minimal realization of H_{V^*} in the sense of Sections 2.1 and 2.2, respectively, where V^* is the adjoint of V as in (1.7).

Lemma 2.4. *The following holds: $T = (T_0')^*$, where $*$ denotes the adjoint of an operator.*

Proof. We need to show that for any $u \in W^{1,2}(E)$ and $f \in W^{-1,2}(E)$, the equation $Tu = f$ is true if and only if

$$(u, T's) = (f, s), \quad \forall s \in C_c^\infty(E), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{1,2}_{\text{loc}}(E)$ and $W^{-1,2}_{\text{comp}}(E)$ extending the inner product in $L^2(E)$ by continuity from $C_c^\infty(E)$.

(1) Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and $Tu = f$. Then $Vu \in W^{-1,2}(E)$. By Lemma 2.2, for all $s \in C_c^\infty(E)$, we have $V^*s \in W^{-1,2}_{\text{comp}}(E)$. Since $s \in C_c^\infty(E)$, we have $V^*s \in L^p_{\text{comp}}(E)$ with p as in Assumption 1.1. By the proof in Remark 1.2, we have $u \in W^{1,2}(E) \subset L^{p'}(E)$ (continuous embedding), where $1/p + 1/p' = 1$. By Hölder's inequality, $L^{p'}_{\text{loc}}(E)$ is in a continuous duality with $L^p_{\text{comp}}(E)$ by the usual integration. Thus, for all $s \in C_c^\infty(E)$, we have (after approximating u by sections $u_j \in C_c^\infty(E)$ in $W^{1,2}$-norm in a neighborhood of $\operatorname{supp} s$)

$$\begin{aligned}(u, V^*s) &= \lim_{j\to\infty}(u_j, V^*s) = \lim_{j\to\infty}\int \langle u_j(x), (V^*s)(x)\rangle\, d\mu(x) \\ &= \int \langle u(x), (V^*s)(x)\rangle\, d\mu(x),\end{aligned} \tag{2.3}$$

where $(\cdot,\cdot)$ is as in (2.2). The second equality in (2.3) holds since $V^*s \in L^1_{\text{loc}}(E)$ by Remark 1.2 and $u_j \in C_c^\infty(E)$.

Therefore, we obtain

$$\begin{aligned}(u, V^*s) &= \int \langle u(x), (V^*s)(x)\rangle\, d\mu(x) \\ &= \int \langle (Vu)(x), s(x)\rangle\, d\mu(x) = (Vu, s),\end{aligned} \tag{2.4}$$

where $(\cdot,\cdot)$ is as in (2.2). The first equality in (2.4) follows from (2.3). The second equality in (2.4) holds by the definition of $(V(x))^* : E_x \to E_x$. The third equality in (2.4) holds for all $s \in C_c^\infty(E)$ since $Vu \in W^{-1,2}(E)$ and $Vu \in L^1_{\text{loc}}(E)$ by Remark 1.2.

Using (2.4), we obtain

$$\begin{aligned}(u, T's) &= (u, \nabla^*\nabla s + V^*s) = (u, \nabla^*\nabla s) + (u, V^*s) \\ &= (\nabla^*\nabla u, s) + (Vu, s) = (Tu, s),\end{aligned} \tag{2.5}$$

where V^* is the adjoint of V as in (1.7) and $(\cdot,\cdot)$ is as in (2.2). In the third equality, we also used the integration by parts (see, e.g., [2, Lemma 8.8]).

(2) Assume that $u \in W^{1,2}(E)$, $f \in W^{-1,2}(E)$, and (2.2) holds. Then the first two equalities in (2.4) hold (we do not know a priori that $Vu \in W^{-1,2}(E)$ so the

third equality in (2.4) is not yet justified). Thus for all $s \in C_c^\infty(E)$,

$$\begin{aligned}(f,s) &= (u,T's) = (u,\nabla^*\nabla s) + (u,V^*s) \\ &= (\nabla^*\nabla u,s) + \int \langle (Vu)(x), s(x)\rangle \, d\mu(x),\end{aligned} \tag{2.6}$$

where the second equality follows as in (2.5), and the third equality follows from integration by parts and the second equality in (2.4).

Since $\nabla^*\nabla u \in W^{-1,2}(E)$ and $f \in W^{-1,2}(E)$, we obtain

$$(f - \nabla^*\nabla u, s) = \int \langle (Vu)(x), s(x)\rangle \, d\mu(x), \quad \forall s \in C_c^\infty(E), \tag{2.7}$$

where $(\cdot,\cdot)$ is as in (2.2).

Since $u \in W^{1,2}(E)$, from Remark 1.2, we know that $Vu \in L^1_{\text{loc}}(E)$. By (2.7), we get $Vu \in W^{-1,2}(E)$ since $C_c^\infty(E)$ is dense in $W^{1,2}(E)$. Thus, as in (2.4),

$$\int \langle (Vu)(x), s(x)\rangle \, d\mu(x) = (Vu,s), \quad \forall s \in C_c^\infty(E), \tag{2.8}$$

where $(\cdot,\cdot)$ is as in (2.2).

From (2.7) and (2.8), we obtain

$$(f - \nabla^*\nabla u, s) = (Vu,s), \quad \forall s \in C_c^\infty(E), \tag{2.9}$$

where $(\cdot,\cdot)$ is as in (2.2).

Therefore,

$$(f,s) = (\nabla^*\nabla u,s) + (Vu,s) = (Tu,s), \quad \forall s \in C_c^\infty(E), \tag{2.10}$$

where $(\cdot,\cdot)$ is as in (2.2).

This shows that $Tu = f$, and the lemma is proven. □

In what follows, we will adopt the terminology of Kato [3] and distinguish between monotone and accretive operators. Accretive operators act within the same Hilbert space, while monotone operators act from a Hilbert space into its adjoint space (antidual).

Lemma 2.5. *The operator T_0 is monotone, that is,*

$$\operatorname{Re}(T_0 s, s) \ge 0, \quad \forall s \in C_c^\infty(E), \tag{2.11}$$

where $(\cdot,\cdot)$ denotes the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$.

Proof. We have for all $s \in C_c^\infty(E)$,

$$\begin{aligned}\operatorname{Re}(T_0 s,s) &= \operatorname{Re}\left[(\nabla^*\nabla s,s) + \int \langle Vs,s\rangle \, d\mu\right] \\ &= \|\nabla s\|^2 + \operatorname{Re}\left[\int \langle V_1 s,s\rangle \, d\mu + i\int \langle V_2 s,s\rangle \, d\mu\right] \\ &\ge \|\nabla s\|^2,\end{aligned} \tag{2.12}$$

where $(\cdot,\cdot)$ is as in (2.11), $\|\cdot\|$ denotes the L^2-norm, and $V_1 \geq 0$ and V_2 are linear selfadjoint bundle endomorphisms as in (1.7).

The lemma is proven. □

Lemma 2.6. *The operator $1+T_0$ is coercive in the sense that*

$$\|(1+T_0)s\|_{-1} \geq \|s\|_1, \quad \forall s \in \operatorname{Dom}(T_0) = C_c^\infty(E), \tag{2.13}$$

where $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$, and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

Proof. As in (2.12), we have for all $s \in C_c^\infty(E)$,

$$\operatorname{Re}((T_0+1)s,s) \geq \|s\|^2 + \|\nabla s\|^2 = \|s\|_1^2, \tag{2.14}$$

where $(\cdot,\cdot)$ is as in (2.11).

Since the left-hand side of (2.14) does not exceed $\|(1+T_0)s\|_{-1}\|s\|_1$, inequality (2.13) immediately follows from (2.14). □

In what follows, Ker A and Ran A denote the kernel and the range of operator A, respectively, and $\bar{A}$ denotes the closure of A.

Lemma 2.7. *The following holds:*

(i) *the operator T_0 is closable with closure T_0^{**},*

(ii) $\operatorname{Ran}(1+T_0^{**})$ *is closed.*

Proof. By Lemma 2.4, it follows that $T' = T_0^*$, where T' is as in Section 2.3. Since $T_0' \subset T'$ (as operators), it follows that T' is densely defined. Thus T_0^{**} exists and equals $\overline{T_0}$. This proves property (i).

We will now prove property (ii). Since $1+T_0$ is coercive by Lemma 2.6, it follows by definition of $\overline{T_0}$ that $1+T_0^{**} = 1+\overline{T_0}$ is also coercive, that is,

$$\|(1+T_0^{**})u\|_{-1} \geq \|u\|_1, \quad \forall u \in \operatorname{Dom}(T_0^{**}), \tag{2.15}$$

where $\|\cdot\|_{-1}$ is the norm in $W^{-1,2}(E)$, and $\|\cdot\|_1$ is the norm in $W^{1,2}(E)$.

We will now show that $\operatorname{Ran}(1+T_0^{**})$ is closed.

Let $f_j \in \operatorname{Ran}(1+T_0^{**})$ and $\|f_j - f\|_{-1} \to 0$ as $j \to \infty$. Let $u_j \in \operatorname{Dom}(1+T_0^{**})$ be a sequence such that $(1+T_0^{**})u_j = f_j$. Since f_j is a Cauchy sequence in $\|\cdot\|_{-1}$, by (2.15) it follows that u_j is a Cauchy sequence in $\|\cdot\|_1$. Thus u_j converges in $\|\cdot\|_1$, and we will denote its limit by u. Since $1+T_0^{**}$ is a closed operator, it follows that $u \in \operatorname{Dom}(1+T_0^{**})$ and $f = (1+T_0^{**})u$. Thus $f \in \operatorname{Ran}(1+T_0^{**})$, and property (ii) is proven. □

In what follows, we will use the general version of Kato's inequality whose proof is given in [2, Theorem 5.7].

Lemma 2.8. *Assume that (M,g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E.*

Assume that $w \in L^1_{\text{loc}}(E)$ *and* $\nabla^*\nabla w \in L^1_{\text{loc}}(E)$. *Then*

$$\Delta_M |w| \le \operatorname{Re}\langle \nabla^*\nabla w, \operatorname{sign} w\rangle, \tag{2.16}$$

where Δ_M *is the scalar Laplacian on* M *and*

$$\operatorname{sign} w(x) = \begin{cases} \dfrac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.17}$$

We now state and prove the key proposition.

Proposition 2.9. *The following holds:* $\operatorname{Ran}(1+T_0^{**}) = W^{-1,2}(E)$.

Proof. By Lemma 2.7, it suffices to show that if $u \in W^{1,2}(E)$ and

$$((1+T_0)s, u) = 0, \quad \forall s \in C_c^\infty(E), \tag{2.18}$$

where $(\cdot,\cdot)$ is as in (2.11), then $u = 0$.

Using condition (2.18) and the same arguments as in the proof of the first two equalities in (2.4) and the equation (2.8), we have

$$\begin{aligned} 0 &= (s,u) + (\nabla^*\nabla s, u) + (Vs, u) \\ &= (s,u) + (s, \nabla^*\nabla u) + (s, V^*u), \quad \forall s \in C_c^\infty(E), \end{aligned} \tag{2.19}$$

where $(\cdot,\cdot)$ is as in (2.11), and V^* is as in (1.7).

Therefore, the following distributional equality holds (recall that by Remark 1.2, we have $V^*u \in L^1_{\text{loc}}(E)$)

$$\nabla^*\nabla u + V^*u + u = 0. \tag{2.20}$$

From (2.20), we have $\nabla^*\nabla u = -V^*u - u \in L^1_{\text{loc}}(E)$. Therefore, by Lemma 2.8, we get

$$\begin{aligned} \Delta_M |u| &\le \operatorname{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle \\ &= \operatorname{Re}\langle -u - V_1 u + iV_2 u, \operatorname{sign} u\rangle \\ &= -|u| - \langle V_1 u, \operatorname{sign} u\rangle \le -|u|, \end{aligned} \tag{2.21}$$

where Δ_M, $\langle\cdot,\cdot\rangle$ and $\operatorname{sign} u$ are as in (2.16), and $V_1 \ge 0$, V_2 are linear selfadjoint bundle endomorphisms as in (1.7).

By (2.21), we get the following distributional inequality:

$$(\Delta_M + 1)|u| \le 0. \tag{2.22}$$

Since (M,g) is a manifold of bounded geometry, by [2, Proposition B.3], inequality (2.22) implies that $|u| = 0$, that is, $u = 0$. This concludes the proof of the proposition. □

Corollary 2.10. *The operator T_0^{**} is maximal monotone (in the sense that it is monotone and has no proper monotone extension).*

Proof. The corollary follows immediately from Proposition 2.9, inequality (2.15), and the remark after equation (3.38) of [4, Section 5.3.10]. □

Proposition 2.11. *The following holds:*
(i) $T = T_0^{**}$,
(ii) *the operator T is maximal monotone.*

Proof. We first prove property (i). Since $T_0 \subset T$ (as operators), it follows that $T_0^{**} \subset T$ because T is closed by Lemma 2.4. By Proposition 2.9, $\operatorname{Ran}(1 + T_0^{**}) = W^{-1,2}(E)$. By the same proposition (with V replaced by V^*), it follows that $\operatorname{Ran}(1 + (T_0')^{**}) = W^{-1,2}(E)$, where T_0' is as in Section 2.3. Since $1 + T = 1 + (T_0')^*$ (see Lemma 2.4), it follows that $\operatorname{Ker}(1+T) = \{0\}$. Hence, T cannot be a proper extension of T_0^{**}. This shows that $T_0^{**} = T$.

Property (ii) follows immediately from property (i) and Corollary 2.10. □

3. Proof of Theorem 1.3.

First, note that the following holds: $u \in \operatorname{Dom}(S)$ if and only if $u \in \operatorname{Dom}(T)$ and $Tu \in L^2(E)$ (in which case $Su = Tu$).

By Propositions 2.9 and 2.11, it follows that $\operatorname{Ran}(1+T) = W^{-1,2}(E)$. Therefore, $\operatorname{Ran}(1 + S) = L^2(E)$. Furthermore, since T is maximal monotone by Proposition 2.11, it follows that

$$\operatorname{Re}(Su, u)_{L^2(E)} = \operatorname{Re}(Tu, u) \geq 0, \quad \forall u \in \operatorname{Dom}(S), \tag{3.1}$$

where $(\cdot,\cdot)_{L^2(E)}$ denotes the inner product in $L^2(E)$, and $(\cdot,\cdot)$ is the duality between $W^{-1,2}(E)$ and $W^{1,2}(E)$.

Thus we proved that S is accretive and $\operatorname{Ran}(1+S) = L^2(E)$. By the remark after equation (3.37) of [4, Section 5.3.10], it follows that S is m-accretive.

References

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[2] M. Braverman, O. Milatovic, and M. Shubin, *Essential self-adjointness of Schrödinger-type operators on manifolds*, Russian Math. Surveys **57** (2002), no. 4, 641–692.
[3] T. Kato, *On some Schrödinger operators with a singular complex potential*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 1, 105–114.
[4] ______, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1980.
[5] M. A. Shubin, *Spectral theory of elliptic operators on noncompact manifolds*, Astérisque (1992), no. 207, 35–108.

Ognjen Milatovic: Department of Mathematics, Fitchburg State College, 160 Pearl Street, Fitchburg, MA 01420, USA
E-mail address: omilatovic@fsc.edu