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# ON THE FRESNEL SINE INTEGRAL AND THE CONVOLUTION

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The Fresnel sine integral $S(x)$, the Fresnel cosine integral $C(x)$, and the associated functions $S_+(x)$, $S_-(x)$, $C_+(x)$, and $C_-(x)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of the Fresnel sine integral and its associated functions with $x_+^r$, $x^r$ are evaluated.

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**1. Introduction.** The Fresnel integrals occur in the diffraction theory and they are of two kinds: the Fresnel integral $S(x)$ with a sine in the integral and the Fresnel integral $C(x)$ with a cosine in the integral.

The *Fresnel sine integral* $S(x)$ is defined by

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin u^2 du \tag{1.1}$$

(see [5]) and the associated functions $S_+(x)$ and $S_-(x)$ are defined by

$$S_+(x) = H(x)S(x), \qquad S_-(x) = H(-x)S(x). \tag{1.2}$$

The *Fresnel cosine integral* $C(x)$ is defined by

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos u^2 du \tag{1.3}$$

(see [5]) and the associated functions $C_+(x)$ and $C_-(x)$ are defined by

$$C_+(x) = H(x)C(x), \qquad C_-(x) = H(-x)C(x), \tag{1.4}$$

where $H$ denotes Heaviside's function.

We define the function $L_r(x)$ by

$$L_r(x) = \int_0^x u^r \sin u^2 du \tag{1.5}$$

for $r = 0, 1, 2, \ldots$. In particular, we have

$$L_0(x) = \sqrt{\frac{\pi}{2}}S(x), \quad L_1(x) = \frac{1}{2} - \frac{1}{2}\cos x^2, \quad L_2(x) = \frac{1}{4}\sqrt{2}\sqrt{\pi}C(x) - \frac{1}{2}(\cos x^2)x. \tag{1.6}$$

We define the functions $\sin_+ x$, $\sin_- x$, $\cos_+ x$, and $\cos_- x$ by

$$\sin_+ x = H(x)\sin x, \qquad \sin_- x = H(-x)\sin x, \qquad \cos_+ x = H(x)\cos x, \qquad \cos_- x = H(-x)\cos x. \tag{1.7}$$

**2. Convolution products.** The classical definition for the convolution product of two functions $f$ and $g$ is as follows.

**Definition 2.1.** Let $f$ and $g$ be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt \tag{2.1}$$

for all points $x$ for which the integral exists.

If the classical convolution $f * g$ of two functions $f$ and $g$ exists, then $g * f$ exists and

$$f * g = g * f. \tag{2.2}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{2.3}$$

The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions $f$ and $g$ in $\mathcal{D}'$ with the following definition, see [4].

**Definition 2.2.** Let $f$ and $g$ be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle \tag{2.4}$$

for arbitrary $\varphi$ in $\mathcal{D}'$, provided that $f$ and $g$ satisfy either of the following conditions:

(a) either $f$ or $g$ has bounded support,

(b) the supports of $f$ and $g$ are bounded on the same side.

It follows that if the convolution $f*g$ exists by this definition, then (2.2) and (2.3) are satisfied.

**Theorem 2.3.** *The convolution* $(\sin_+ x^2)*x_+^r$ *exists and*

$$(\sin_+ x^2)*x_+^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_{r-i}(x)x_+^i \tag{2.5}$$

*for* $r=0,1,2,\dots$.

**Proof.** It is obvious that $(\sin_+ x^2)*x_+^r = 0$ if $x<0$. When $x>0$, we have

$$\begin{aligned}(\sin_+ x^2)*x_+^r &= \int_0^x \sin t^2 (x-t)^r\,dt \\ &= \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}L_{r-i}(x)x^i,\end{aligned} \tag{2.6}$$

thus proving (2.5). □

**Corollary 2.4.** *The convolution* $(\sin_- x^2)*x_-^r$ *exists and*

$$(\sin_- x^2)*x_-^r = \sum_{i=0}^{r}\binom{r}{i}L_{r-i}(x)x_-^i \tag{2.7}$$

*for* $r=0,1,2,\dots$.

**Proof.** Equation (2.7) follows on replacing $x$ by $-x$ in (2.5) and noting that

$$L_r(-x) = (-1)^{r+1}L_r(x). \tag{2.8}$$

□

**Theorem 2.5.** *The convolution* $S_+(x)*x_+^r$ *exists and*

$$S_+(x)*x_+^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}L_{r-i+1}(x)x_+^i \tag{2.9}$$

*for* $r=0,1,2,\dots$.

**Proof.** It is obvious that $S_+(x)*x_+^r=0$ if $x<0$. When $x>0$, we have

$$\begin{aligned}\sqrt{\frac{\pi}{2}}S_+(x)*x_+^r &= \int_0^x (x-t)^r\int_0^t \sin u^2\,du\,dt \\ &= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}L_{r-i+1}(x)x_+^i.\end{aligned} \tag{2.10}$$

Thus equation (2.9) follows. □

**Corollary 2.6.** *The convolution $S_-(x) * x_-^r$ exists and*

$$S_-(x) * x_-^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r+1} \binom{r+1}{i} L_{r-i+1}(x) x_-^i \tag{2.11}$$

*for* $r = 0, 1, 2, \ldots$.

**Proof.** Equation (2.11) follows on replacing $x$ by $-x$ in (2.9). □

**3. Existence of neutrix convolution product.** In order to extend the convolution product to a larger class of distributions, the neutrix convolution product was introduced in [1] and was later extended in [2, 3]. For the further extension, first of all, we let $\tau$ be a function in $\mathcal{D}$ having the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le 1/2$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function $\tau_\nu$ is now defined for $\nu > 0$ by

$$\tau_\nu(x) = \begin{cases} 1, & |x| \le \nu, \\ \tau(\nu^\nu x - \nu^{\nu+1}), & x > \nu, \\ \tau(\nu^\nu x + \nu^{\nu+1}), & x < -\nu. \end{cases} \tag{3.1}$$

**Definition 3.1.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $f_\nu = f\tau_\nu$ for $\nu > 0$. The neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_\nu * g\}$, provided that the limit $h$ exists in the sense that

$$\underset{\nu\to\infty}{N\text{-}\lim} \langle f_\nu * g, \varphi \rangle = \langle h, \varphi \rangle, \tag{3.2}$$

for all $\varphi$ in $\mathcal{D}$, where $N$ is the neutrix, see van der Corput [7], having domain $N'$, the positive real numbers, with negligible functions finite linear sums of the functions $\nu^\lambda \ln^{r-1} \nu$, $\ln^r \nu$, $\nu^r \sin \nu^2$, and $\nu^r \sin \nu^2$ $(\lambda \ne 0,\ r = 1, 2, \ldots)$ and all functions which converge to zero in the normal sense as $\nu$ tends to infinity.

Note that in this definition the convolution product $f_\nu * g$ is defined in Gel'fand and Shilov's sense, with the distribution $f_\nu$ having bounded support.

It was proved in [1] that if $f * g$ exists in the classical sense or by Definition 2.1, then $f \circledast g$ exists and

$$f \circledast g = f * g. \tag{3.3}$$

The following theorem was also proved in [1].

**Theorem 3.2.** *Let $f$ and $g$ be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$*

*exists and*

$$(f \circledast g)' = f \circledast g'. \tag{3.4}$$

Now if we let $L_r = N\text{-}\lim_{\nu\to\infty} L_r(\nu)$ and note that

$$S(\infty) = C(\infty) = \frac{1}{2}, \tag{3.5}$$

see Olver [6], then we have the following theorem.

**Theorem 3.3.** *The neutrix convolution* $(\sin_+ x^2) * x^r$ *exists and*

$$(\sin_+ x^2) \circledast x^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i} x^i \tag{3.6}$$

*for* $r = 0, 1, 2, \ldots$.

**Proof.** We set

$$(\sin_+ x^2)_\nu = (\sin_+ x^2)\tau_\nu(x). \tag{3.7}$$

Then the convolution $(\sin_+ x^2)_\nu * x^r$ exists and

$$(\sin_+ x^2)_\nu * x^r = \int_0^\nu \sin t^2 (x-t)^r\,dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \sin t^2 (x-t)^r\,dt. \tag{3.8}$$

Now

$$\begin{aligned} \int_0^\nu \sin t^2 (x-t)^r\,dt &= \sum_{i=0}^{r} \binom{r}{i} \int_0^\nu x^i (-t)^{r-i} \sin t^2\,dt \\ &= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i}(\nu) x^i, \end{aligned} \tag{3.9}$$

and it follows that

$$N\text{-}\lim_{\nu\to\infty} \int_0^\nu \sin t^2 (x-t)^r\,dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} L_{r-i} x^i. \tag{3.10}$$

Further, it can easily be seen that for each fixed $x$,

$$\lim_{\nu\to\infty} \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t) \sin t^2 (x-t)^r\,dt = 0, \tag{3.11}$$

and (3.6) follows from (3.9), (3.10), and (3.11). □

**Theorem 3.4.** *The neutrix convolution* $S_+(x) \circledast x^r$ *exists and*

$$S_+(x) \circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} L_{r-i+1} x^i \tag{3.12}$$

*for* $r = 0, 1, 2, \ldots$.

**Proof.** We put $[S_+(x)]_\nu = S_+(x)\tau_\nu(x)$. Then the convolution product $[S_+(x)]_\nu * x^r$ exists and

$$[S_+(x)]_\nu * x^r = \int_0^\nu S(t)(x-t)^r dt + \int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t)S(t)(x-t)^r dt. \tag{3.13}$$

We have

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\int_0^\nu S(t)(x-t)^r dt &= \int_0^\nu (x-t)^r \int_0^t \sin u^2 du\, dt \\ &= -\frac{1}{r+1}\int_0^\nu \sum_{i=0}^{r}\binom{r+1}{i} x^i\left[(-\nu)^{r-i+1}-(-u)^{r-i+1}\right]\sin u^2 du,\end{aligned} \tag{3.14}$$

and it follows that

$$N\text{-}\lim_{\nu\to\infty}\int_0^\nu S(t)(x-t)^r dt = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i+1}L_{r-i+1}x^i. \tag{3.15}$$

Further, it is easily seen that for each fixed $x$,

$$\lim_{\nu\to\infty}\int_\nu^{\nu+\nu^{-\nu}} \tau_\nu(t)S(t)(x-t)^r dt = 0, \tag{3.16}$$

and (3.12) now follows immediately from (3.14), (3.15), and (3.16). □

**Corollary 3.5.** *The neutrix convolution $S_-(x) \circledast x^r$ exists and*

$$S_-(x)\circledast x^r = \frac{\sqrt{2}}{\sqrt{\pi}(r+1)}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}L_{r-i+1}x^i \tag{3.17}$$

*for* $r = 0,1,2,\dots$.

**Proof.** Equation (3.17) follows on replacing $x$ by $-x$ and $L_r$ by $(-1)^{r+1}L_r$ in (3.12). □

**Corollary 3.6.** *The neutrix convolution $S(x)\circledast x^r$ exists and*

$$S(x)\circledast x^r = 0 \tag{3.18}$$

*for* $r = 0,1,2,\dots$.

**Proof.** Equation (3.18) follows from (3.12) and (3.17) on noting that $S(x) = S_+(x)+S_-(x)$. □

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