DERIVATIONS ON BANACH ALGEBRAS

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Let $D$ be a derivation on a Banach algebra; by using the operator $D^2$, we give necessary and sufficient conditions for the separating ideal of $D$ to be nilpotent. We also introduce an ideal $M(D)$ and apply it to find out more equivalent conditions for the continuity of $D$ and for nilpotency of its separating ideal.

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1. Introduction. Let $A$ be a Banach algebra. By a derivation on $A$, we mean a linear mapping $D : A \to A$, which satisfies $D(ab) = aD(b) + D(a)b$ for all $a$ and $b$ in $A$. The separating space of $D$ is the set

$$S(D) = \{a \in A : \exists \{a_n\} \subset A; a_n \to 0, D(a_n) \to a\}.$$  \hfill (1.1)

The set $S(D)$ is a closed ideal of $A$ which, by the closed-graph theorem, is zero if and only if $D$ is continuous.

**Definition 1.1.** A closed ideal $J$ of $A$ is said to be a separating ideal if, for each sequence $\{a_n\}$ in $A$, there is a natural $N$ such that

$$\overline{(Ja_n \cdots a_1)} = \overline{(Ja_N \cdots a_1)} \quad (n \geq N).$$ \hfill (1.2)

The separating space of a derivation on $A$ is a separating ideal [2, Chapter 5]; it also satisfies the same property for the left products.

The following assertions are of the most famous conjectures about derivations on Banach algebras:

- (C1) every derivation on a Banach algebra has a nilpotent separating ideal;
- (C2) every derivation on a semiprime Banach algebra is continuous;
- (C3) every derivation on a prime Banach algebra is continuous;
- (C4) every derivation on a Banach algebra leaves each primitive ideal invariant.

Clearly, if (C1) is true, then the same for (C2) and (C3). Mathieu and Runde in [5] proved that (C1), (C2), and (C3) are equivalent. The conjecture (C4) is known as the noncommutative Singer-Wermer conjecture, and it has been proved in [1] that if each of the conjectures (C1), (C2), or (C3) hold, then (C4) is also true. The conjectures (C1), (C2), and (C3) are still open even if $A$ is assumed
to be commutative, but (C4) is true in the commutative case, see [7]. These conjectures are also related to some other famous open problems; the reader is referred to [1, 3, 4, 5, 9] for more details.

In the next section, we deal with (C1), and although, for a derivation $D$ on a Banach algebra, the operators $D^n$, $n = 2, 3, \ldots$, are more complicated, by considering $D^2$, we easily give some equivalent conditions for $S(D)$ to be nilpotent. As a consequence, we reprove some of the results in [8]. At the end of the next section, we introduce an ideal related to a derivation and apply it to obtain some equivalent conditions for continuity of $D$ and for nilpotency of $S(D)$.

We recall that $S(D)$ is nilpotent if and only if $S(D) \cap R$ is nilpotent, see [1, Lemma 4.2].

2. The results. From now on, $A$ is a Banach algebra, and $R$ and $L$ denote the Jacobson radical and the nil radical of $A$, respectively, (see [6, Chapter 4] for definitions). Note that $D$ is a derivation on $A$, and $S(D)$ is the separating ideal of $D$. If $B_i$’s, $i = 1, 2, \ldots, n$, are subsets of $A$, then $B_1B_2 \cdots B_n$ denotes the linear span of the set \{ $b_1b_2 \cdots b_n : b_i \in B_i$, for $i = 1, 2, \ldots, n$ \}, and if all of $B_i$’s coincide with each other, we denote this set by $B^n$.

**Theorem 2.1.** Let $J$ be a closed left ideal of $A$. Then, $S(D) \cap J$ is nilpotent if and only if $D^2 \mid \bigcap_{n=1}^{\infty} (S(D) \cap J)^n$ is continuous.

**Proof.** Suppose that $D^2$ is continuous on $\bigcap_{n=1}^{\infty} (S(D) \cap J)^n$. Consider $a$ in $S(D) \cap J$, then for each $n \in \mathbb{N}$, $a^n \in (S(D) \cap J)^n$, and since $S(D)$ is a separating ideal, there exists $N \in \mathbb{N}$ such that

\[ S(D)a^n = S(D)a^N \quad (n \geq N). \quad (2.1) \]

Hence, by the Mittag-Leffler theorem [2, Theorem A.1.25] and the fact that $S(D)a^n \subseteq (S(D) \cap J)^n$, we have

\[ \overline{S(D)a^N} = \bigcap_{n=1}^{\infty} S(D)a^n = \bigcap_{n=1}^{\infty} (S(D) \cap J)^n. \quad (2.2) \]

Now, let \{ $x_n$ \} $\subseteq A$, $x_n \rightarrow 0$, and $D(x_n) \rightarrow a^{N+1}$. Take $y_n = x_n a^{N+1}$, then $y_n \in S(D)a^N \subseteq \overline{S(D) \cap J}^n$, $y_n \rightarrow 0$, and $D(y_n) \rightarrow a^{2(N+1)}$, and by the hypothesis, $D^2(y_n) \rightarrow 0$ and $D^2(y_n^2) \rightarrow 0$. On the other hand,

\[ D^2(y_n^2) = y_n D^2(y_n) + 2(Dy_n)^2 + D^2(y_n)y_n \rightarrow 2a^{4(N+1)}. \quad (2.3) \]

Therefore, $a^{4N+4} = 0$, that is, $S(D) \cap J$ is a nil and hence a nilpotent ideal by closedness [6, Theorem 4.4.11]. The converse is trivial. \[ \square \]
Remark 2.2. (i) Note that in Theorem 2.1, we can replace \( J \) by a right ideal, see [2, Theorem 5.2.24].

(ii) The argument of Theorem 2.1 shows that if \( J \) is not assumed to be closed and if \( D^2 \) is continuous on \( \bigcap_{n=1}^{\infty} (S(D) \cap J)^n \), then \( S(D) \cap J \) will be a nil ideal.

Corollary 2.3. The set \( S(D) \) is nilpotent if and only if \( D^2 \big|_{\bigcap_{n=1}^{\infty} (S(D) \cap R)^n} \) is continuous.

Proof. If \( S(D) \) is nilpotent, then the result is obvious. Conversely, by Theorem 2.1, \( S(D) \cap R \) is nilpotent, and by [1, Lemma 4.2], \( S(D) \) is nilpotent.

Corollary 2.4. If \( \dim(\bigcap_{n=1}^{\infty} (S(D) \cap R)^n) < \infty \), then \( S(D) \) is nilpotent.

The assertions of the following theorem were proved by Villena in [8], see also [9, Theorem 4.4]. Using Theorem 2.1, we can reprove them in a different way.

Theorem 2.5. The derivation \( D \) is continuous if one of the following assertions hold:

(a) \( A \) is semiprime and \( \dim(R \cap (\bigcap_{n=1}^{\infty} A^n)) < \infty \);
(b) \( A \) is prime and \( \dim(\bigcap_{n=1}^{\infty} (aA \cap R)^n) < \infty \) for some \( a \in A \) with \( a^2 \neq 0 \);
(c) \( A \) is an integral domain and \( \dim(\bigcap_{n=1}^{\infty} (aA \cap R)^n) < \infty \) for some nonzero \( a \in A \).

Proof. (a) By Corollary 2.4, \( S(D) \) is nilpotent, and since \( A \) is semiprime, \( D \) is continuous.

(b) Without loss of generality, we may assume that \( A \) has an identity. By assumption, \( \bigcap_{n=1}^{\infty} (aA \cap R \cap S(D))^n \) is finite dimensional; thus, \( D^2 \) is continuous on this space, and by Remark 2.2(ii), \( aA \cap R \cap S(D) \) is a nil right ideal; therefore, \( a(S(D) \cap R) \) is a nil right ideal, and by [6, Theorem 4.4.11], \( a(S(D) \cap R) \subseteq L = \{0\} \). Thus, \( aAaS(D) \cap R) = \{0\} \), where \( aAa \) is the ideal generated by \( a \). Since \( a^2 \neq 0 \) and \( A \) is prime, it follows that \( S(D) \cap R = \{0\} \) and hence \( S(D) \subseteq L = \{0\} \).

(c) The same argument as in (b) shows that \( a(S(D) \cap R) = \{0\} \), and since \( A \) is an integral domain, \( S(D) \cap R = \{0\} \) and \( D \) is continuous.

In the sequel, we give other equivalent conditions for \( S(D) \) to be nilpotent, but first we introduce the set

\[
M(D) = \{ x \in S(D) \cap R : D(x) \in R \}.
\] (2.4)

Obviously, \( M(D) \) is an ideal of \( A \) and \( (S(D) \cap R)^2 \subseteq M(D) \). The following theorems show that this ideal can help us to study the continuity of a derivation or nilpotency of its separating ideal.

Theorem 2.6. The derivation \( D \) is continuous if and only if \( M(D) = \{0\} \).
**Proof.** Clearly, if $D$ is continuous, then $M(D) = \{0\}$. Conversely, let $M(D) = \{0\}$; then, $(S(D) \cap R)^2 = \{0\}$. Therefore, $(S(D) \cap R)$ and hence $S(D)$ is a nilpotent ideal. Therefore, $S(D) \subseteq I$; we also have $D(L) \subseteq L$ by [1, Lemma 4.1]; thus, $D(S(D)) \subseteq R$, that is, $S(D) \subseteq M(D) = \{0\}$ and $D$ is continuous.

**Theorem 2.7.** The following assertions are equivalent:

(a) $S(D)$ is nilpotent;
(b) $M(D)$ is a nil ideal;
(c) $\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$.

**Proof.** Clearly, (a) implies (b). Suppose that (b) holds, then $(S(D) \cap R)^2$ is a nil ideal; therefore, $S(D)$ is a nilpotent ideal and (a) holds. Now, if $S(D)$ is nilpotent, then $\bigcap_{n=1}^{\infty} (S(D))^n = \{0\}$ and this implies (c). Finally, if $\bigcap_{n=1}^{\infty} M(D)^n = \{0\}$, then by Theorem 2.1 and Remark 2.2 $M(D) = M(D) \cap S(D)$ is a nil ideal and (c) implies (b).

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**References**


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Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

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