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L^∞-ERROR ESTIMATE FOR A SYSTEM OF ELLIPTIC QUASIVARIATIONAL INEQUALITIES

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We deal with the numerical analysis of a system of elliptic quasivariational inequalities (QVIs). Under $W^{2,p}(\Omega)$-regularity of the continuous solution, a quasi-optimal L^∞-convergence of a piecewise linear finite element method is established, involving a monotone algorithm of Bensoussan-Lions type and standard uniform error estimates known for elliptic variational inequalities (VIs).

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1. Introduction. In this paper, we are concerned with the L^∞-convergence of the standard finite element approximation for the following system of quasivariational inequalities (QVIs): find $U = (u^1,\dots,u^M) \in (H_0^1(\Omega))^J$ satisfying

$$\begin{gathered} a^i(u^i, v-u^i) \geqq (f^i, v-u^i) \quad \forall v \in H_0^1(\Omega), \\ u^i \leq Mu^i, \quad u^i \geq 0, \quad v \leq Mu^i, \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$, $a^i(u,v)$ are J-elliptic bilinear forms continuous on $H^1(\Omega)\times H^1(\Omega)$, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and f^i are J-regular functions.

This system, introduced by Bensoussan and Lions (see [3]), arises in the management of energy production problems where J-units are involved (see [4] and the references therein). In the case studied here, Mu^i represents a "cost function" and the prototype encountered is

$$Mu^i(x) = k + \inf_{\mu\neq i} u^\mu(x), \tag{1.2}$$

where k represents the switching cost. It is positive when the unit is "turn on" and equal to zero when the unit is "turn off."

Note also that the operator M provides the coupling between the unknowns $u^1,\dots,u^J$.

Naturally, the structure of problem (1.1) is analogous to that of the classical obstacle problem where the obstacle is replaced by an implicit one depending upon the solution sought. The terminology QVI being chosen is a result of this remark.

The L^∞-error estimate is a challenge not only for its practical reasons but also due to its inherent difficulty of convergence in this norm. Moreover, the interest in using such a norm for the approximation of obstacle problems is that they are a type of free boundary problems. This fact has been validated by the paper of Brezzi and Caffarelli [7] and later by that of Nochetto [15] on the convergence of the discrete free boundary to the continuous one.

A lot of results on error estimates for the classical obstacle problems and variational inequalities (VIs) were achieved in this norm, (cf. [1, 11, 14, 16]). However, very few works are known on this subject concerning QVIs (cf. [5, 10]) and especially the case of systems (see [6]).

Our primary aim in this paper is, precisely, to show that problem (1.1) can be properly approximated by a finite element method which turns out to be quasi-optimally accurate in $L^\infty(\Omega)$. The approximation is carried out by first introducing a monotone iterative scheme of Bensoussan-Lions type which is shown to converge geometrically to the continuous solution. Similarly, using the standard finite element method and a discrete maximum principle (d.m.p.), the solution of the discrete system of QVIs is in its turn approximated by an analogue discrete monotone iterative scheme, and a geometric convergence to the discrete solution is given as well. An L^∞-error estimate is then established combining the geometric convergence of both the continuous and discrete iterative schemes with known uniform error estimates in elliptic VIs.

An outline of the paper is as follows. We lay down some necessary notations, assumptions, and preliminaries in Section 2. We consider the continuous problem and prove some related qualitative properties in Section 3. Section 4 deals with the discrete problem for which an analogue study to that of the continuous problem is achieved. Finally, in Section 5, we prove a fundamental lemma and give the main result.

2. Preliminaries

2.1. Assumptions and notation. We are given functions $a^i_{jk}(x)$, $a^i_k(x)$, and $a^i_0(x)$, $1 \le i \le J$, sufficiently smooth such that

$$\sum_{1\le j,\,k\le N} a^i_{jk}(x)\xi_j\xi_k \geqq \alpha|\zeta|^2, \quad \zeta \in \mathbb{R}^N,\ \alpha > 0, \tag{2.1}$$

$$a^i_0(x) \geqq \beta > 0, \quad x \in \Omega. \tag{2.2}$$

We define the variational forms, for any $u, v \in H^1(\Omega)$,

$$a^i(u,v) = \int_\Omega \left(\sum_{1\le j,\,k\le N} a^i_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} a^i_k(x)\frac{\partial u}{\partial x_k}v + a^i_0(x)uv \right) dx \tag{2.3}$$

such that

$$a^i(v,v) \geqq \gamma \|v\|^2_{H^1(\Omega)}, \quad \gamma > 0, \tag{2.4}$$

and the differential operator associated with the bilinear form $a^i(\cdot,\cdot)$

$$\mathcal{A}^i = -\sum_{1\le j,k\le N} \frac{\partial}{\partial x_j} a^i_{jk}(x) \frac{\partial}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x) \frac{\partial}{\partial x_k} + a^i_0(x). \tag{2.5}$$

We are also given right-hand sides

$$f^1,\dots,f^J \quad \text{such that } f^i \in L^\infty(\Omega),\ f^i \ge 0. \tag{2.6}$$

2.2. Elliptic VIs

Definition 2.1. Let $f \in L^\infty(\Omega)$ and $\psi \in W^{1,\infty}(\Omega)$ such that $\psi \ge 0$ on $\partial\Omega$. The following problem is called an elliptic VI: find $u \in \mathbb{K}$ such that

$$a(u,v-u) \geqq (f,v-u) \quad \forall v \in \mathbb{K}, \tag{2.7}$$

where $\mathbb{K} = \{v \in H^1_0(\Omega)$ such that $v \le \psi$ a.e.$\}$ and $a(\cdot,\cdot)$ is a bilinear form of the same type as those defined in (2.3).

2.2.1. Levy-Stampacchia inequality

Lemma 2.2 (cf. [2, 3]). *Let $\psi \in H^1(\Omega)$ such that $\psi \ge 0$ on $\partial\Omega$. Let also $\mathcal{A}$ be the differential operator associated with the bilinear form $a(\cdot,\cdot)$ and u be the solution of VI (2.7) such that $\mathcal{A}u \ge g$ (in the sense of $H^{-1}(\Omega)$), where $g \in L^2(\Omega)$. Then*

$$f \ge \mathcal{A}\psi \ge f \wedge g. \tag{2.8}$$

Theorem 2.3 (cf. [2, 13]). *Under the conditions of Lemma 2.2, the solution u of (2.7) satisfies the property $u \in W^{2,p}(\Omega)$ for all $p \ge 2$, $p < \infty$, $\mathcal{A}u \in L^\infty(\Omega)$.*

2.2.2. A monotonicity property

Theorem 2.4 (cf. [2]). *Let (f,ψ) and $(\tilde{f},\tilde{\psi})$ be a pair of data and $u = \sigma(f,\psi)$ and let $\tilde{u} = (\tilde{f},\tilde{\psi})$ be the respective solutions of (2.7). If $f \ge \tilde{f}$ and $\psi \ge \tilde{\psi}$, then $\sigma(f,\psi) \ge \sigma(\tilde{f},\tilde{\psi})$.*

From now on, we will adopt the notation $\sigma(\psi)$ instead of $\sigma(f,\psi)$.

Proposition 2.5 (cf. [12]). *The mapping σ is increasing and concave with respect to ψ.*

The following proposition plays an important role in proving Proposition 3.4.

Proposition 2.6. *Let c be a positive constant. Then $\sigma(\psi+c) \leq \sigma(\psi)+c$.*

Proof. Clearly $\sigma(\psi)+c = u+c$ is solution to the VI with right-hand side $f+a_0c$ and obstacle $\psi+c$ whereas $\sigma(\psi+c)$ is solution to the VI with right-hand side f and obstacle $\psi+c$. Then, as $a_0(x) \geq \beta > 0$ (see (2.2)) and $c > 0$, it follows that $f < f+a_0c$ and thanks to Theorem 2.4 we get $\sigma(\psi+c) \leq \sigma(\psi)+c$. □

3. The continuous problem

3.1. Existence, uniqueness, and regularity. The existence of a unique solution to system (1.1) can be proved adapting the approach developed in [3, pages 343–358].

Let $L^\infty_+(\Omega)$ denote the positive cone of $L^\infty(\Omega)$, and consider $\mathbb{H}^+ = (L^\infty_+(\Omega))^J$ equipped with the norm

$$\|V\|_\infty = \max_{1\leq i\leq J} ||v^i||_{L^\infty(\Omega)}. \tag{3.1}$$

We define the following fixed-point mapping:

$$\begin{gathered} T : \mathbb{H}^+ \longrightarrow \mathbb{H}^+, \\ W \longrightarrow TW = \zeta = (\zeta^1,\ldots,\zeta^J), \end{gathered} \tag{3.2}$$

where $\zeta^i = \sigma(Mw^i) \in H^1_0(\Omega)$ is a solution to the following VI:

$$\begin{gathered} a^i(\zeta^i, v-\zeta^i) \geqq (f^i, v-\zeta^i) \quad \forall v \in H^1_0(\Omega), \\ \zeta^i \leq Mw^i, \quad v \leq Mw^i. \end{gathered} \tag{3.3}$$

Problem (3.3) being a coercive VI, thanks to [2, 13] it has one and only one solution.

Consider now $\overline{U}^0 = (\overline{u}^{1,0},\ldots,\overline{u}^{J,0})$, where $\overline{u}^{i,0}$ is solution to the following variational equation:

$$a^i(\overline{u}^{i,0}, v) = (f^i, v) \quad \forall v \in H^1_0(\Omega). \tag{3.4}$$

Due to [3], problem (3.4) has a unique solution. Moreover, $\overline{u}^{i,0} \in W^{2,p}(\Omega)$, $2 \leq p < \infty$.

3.1.1. Some properties of the mapping T. The mapping T possesses the following properties.

Proposition 3.1. *Let $\mathbb{C} = \{W \in \mathbb{H}^+$ such that $0 \leq W \leq \overline{U}^0\}$. Then T maps $\mathbb{C}$ into itself.*

Proof. (1) $TW \le \overline{U}^0$ for all $W \in \mathbb{H}^+$.

For all $\varphi \in H^1(\Omega)$, we let $\varphi^+ = \max(\varphi, 0)$. By the fact that both of ζ^i and $\overline{u}^{i,0}$ belong to $H_0^1(\Omega)$, we clearly have

$$\zeta^i - (\zeta^i - \overline{u}^{i,0})^+ \in H_0^1(\Omega). \tag{3.5}$$

Moreover, as $(\zeta^i - \overline{u}^{i,0})^+ \ge 0$, it follows that

$$\zeta^i - (\zeta^i - \overline{u}^{i,0})^+ \le \zeta^i \le M w^i. \tag{3.6}$$

Therefore, we can take $v = \zeta^i - (\zeta^i - \overline{u}^{i,0})^+$ as a trial function in (3.3). This gives

$$a^i\Big(\zeta^i, -(\zeta^i - \overline{u}^{i,0})^+\Big) \geqq \Big(f^i, -(\zeta^i - \overline{u}^{i,0})^+\Big). \tag{3.7}$$

Also, for $v = (\zeta^i - \overline{u}^{i,0})^+$, (3.4) becomes

$$a\Big(\overline{u}^{i,0}, (\zeta^i - \overline{u}^{i,0})^+\Big) = \Big(f^i, (\zeta^i - \overline{u}^{i,0})^+\Big). \tag{3.8}$$

So, by addition, we obtain

$$-a^i\Big((\zeta^i - \overline{u}^{i,0})^+, (\zeta^i - \overline{u}^{i,0})^+\Big) \geqq 0, \tag{3.9}$$

which, by (2.4), yields

$$(\zeta^i - \overline{u}^{i,0})^+ = 0; \tag{3.10}$$

thus

$$\zeta^i \le \overline{u}^{i,0} \quad \forall i = 1, 2, \dots, J, \tag{3.11}$$

that is,

$$TW \le \overline{U}^0. \tag{3.12}$$

(2) $TW \ge 0$, for all $W \in \mathbb{H}^+$.

This follows immediately from standard comparison results in elliptic VIs since $f^i \ge 0$. □

Proposition 3.2. *The mapping T is increasing on $\mathbb{H}^+$.*

Proof. It follows immediately from the increasing property of the mapping σ (see Proposition 2.5). □

Proposition 3.3. *The mapping T is concave on $\mathbb{H}^+$.*

Proof. It follows immediately from the concaveness of the mapping σ (see Proposition 2.5). □

Proposition 3.4. *The mapping T is Lipschitz continuous on $\mathbb{H}^+$, that is,*

$$\|TW - T\tilde{W}\|_\infty \le \|W - \tilde{W}\|_\infty \quad \forall W, \tilde{W} \in \mathbb{H}^+. \tag{3.13}$$

Proof. Let $W = (w^1, \dots, w^J)$, $\tilde{W} = (\tilde{w}^1, \dots, \tilde{w}^J)$, and $\delta = (\delta^1, \dots, \delta^J)$ such that

$$\delta^i = ||w^i - \tilde{w}^i||_{L^\infty(\Omega)}. \tag{3.14}$$

Now, setting

$$\Phi = \|\delta\|_\infty, \tag{3.15}$$

the monotonicity property of T implies that

$$\begin{aligned} TW &\le T(\tilde{W} + \delta) \\ &\le \left(\sigma(M(\delta^1 + \tilde{w}^1)), \dots, \sigma(M(\delta^i + \tilde{w}^i)), \dots, \sigma(M(\delta^M + \tilde{w}^J))\right) \\ &= \left(\sigma(\delta^1 + M\tilde{w}^1), \dots, \sigma(\delta^i + M\tilde{w}^i), \dots, \sigma(\delta^M + M\tilde{w}^J)\right) \\ &\le \left(\sigma(M\tilde{w}^1) + \delta^1, \dots, \sigma(M\tilde{w}^i) + \delta^i, \dots, \sigma(M\tilde{w}^M) + \delta^J\right) \end{aligned} \tag{3.16}$$

due to Proposition 2.6. Thus

$$TW \le T\tilde{W} + \delta. \tag{3.17}$$

Interchanging the roles of W and $\tilde{W}$, one can similarly get

$$T\tilde{W} \le TW + \delta. \tag{3.18}$$

This completes the proof. □

Remark 3.5. The discrete version of Proposition 3.4 plays an important role in the finite element error analysis part of this work.

Remark 3.6. We notice that the solutions of system (1.1) correspond to fixed points of mapping T, that is, $U = TU$. Then, in this view, it is natural to consider the following iterative scheme.

3.1.2. A continuous iterative scheme of Bensoussan-Lions type. Starting from $\overline{U}^0$ defined in (3.4) and $\underline{U}^0 = (0,\dots,0)$, we define the sequences

$$\overline{U}^{n+1} = T\overline{U}^n, \quad n = 0,1,\dots, \tag{3.19}$$

$$\underline{U}^{n+1} = T\underline{U}^n, \quad n = 0,1,\dots. \tag{3.20}$$

The convergence analysis of these sequences rests upon the following results.

Lemma 3.7. *Let $0 < \lambda < \inf(k/\|\overline{U}^0\|_\infty, 1)$. Then $T(0) \geqq \lambda\overline{U}^0$.*

Proof. The proof is very similar to that of [3, page 351]. □

Proposition 3.8. *Let $\gamma \in\,]0,1]$ and $W, \tilde{W} \in \mathbb{C}$ such that*

$$W - \tilde{W} \le \gamma W. \tag{3.21}$$

Then, under the conditions of Lemma 3.7,

$$TW - T\tilde{W} \le \gamma(1-\lambda)TW. \tag{3.22}$$

Proof. From (3.21), we have $(1-\gamma)W \le \tilde{W}$. Then, applying Proposition 3.3, we get

$$(1-\gamma)TW + \gamma T(0) \le T[(1-\gamma)W + \gamma\cdot 0] \le T\tilde{W}, \tag{3.23}$$

and, due to Lemma 3.7, the desired result follows. □

3.1.3. Convergence of the continuous iterative scheme

Theorem 3.9. *Under conditions of Propositions 3.1, 3.2, 3.3, and 3.8, the sequences $(\overline{U}^n)$ and $(\underline{U}^n)$ are monotone and well defined in $\mathbb{C}$. Moreover, they converge, respectively, from above and below to the unique solution of system (1.1).*

Proof. It is an adaptation of [3, pages 342–358]. □

3.1.4. Regularity of the solution of system (1.1)

Theorem 3.10 [3, page 453]. *Assume $a^i_{jk}(x)$ in $C^{1,\alpha}(\overline{\Omega})$, $a^i(x)$, and $a^i_0(x)$ and f^i in $C^{0,\alpha}(\overline{\Omega})$, $\alpha > 0$. Then, $(u^1,\dots,u^M) \in (W^{2,p}(\Omega))^J$, $2 \le p < \infty$.*

3.2. Rate of convergence of the continuous iterative scheme

Proposition 3.11. *Let the conditions of Proposition 3.8 hold. Then*

$$\|\overline{U}^n - U\|_\infty \le (1-\lambda)^n \|\overline{U}^0\|_\infty, \tag{3.24}$$

$$\|\underline{U}^n - U\|_\infty \le (1-\lambda)^n \|\overline{U}^0\|_\infty. \tag{3.25}$$

Proof. By Theorem 3.9, we have

$$0 \le U \le \overline{U}^0, \tag{3.26}$$

so

$$0 \le \overline{U}^0 - U \le \overline{U}^0. \tag{3.27}$$

Then, applying (3.21) and (3.22) with $\gamma = 1$, we get

$$0 \le T\overline{U}^0 - TU \le (1-\lambda)T\overline{U}^0 \tag{3.28}$$

and by (3.19),

$$0 \le \overline{U}^1 - U \le (1-\lambda)\overline{U}^1. \tag{3.29}$$

Now, using (3.21) and (3.22) again with $\gamma = 1-\lambda$, it follows that

$$0 \le T\overline{U}^1 - TU \le (1-\lambda)(1-\lambda)T\overline{U}^1, \tag{3.30}$$

that is,

$$0 \le \overline{U}^2 - U \le (1-\lambda)^2\overline{U}^2 \tag{3.31}$$

and inductively,

$$0 \le \overline{U}^n - U \le (1-\lambda)\overline{U}^n \le (1-\lambda)^n\overline{U}^0. \tag{3.32}$$

We prove estimation (3.25) as estimation (3.24). □

4. The discrete problem. Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform.

Let $\mathbb{V}_h$ denote the standard piecewise linear finite element space and $\mathbb{A}^i$, $1 \le i \le J$, be the matrices with generic coefficients $a^i(\varphi_l, \varphi_s)$, where φ_s, $s = 1, 2, \ldots, m(h)$, are the nodal basis functions. Let also r_h be the usual interpolation operator.

The d.m.p. We assume that $\mathbb{A}^i$ are M-matrices (cf. [9]).

Let $u_h \in \mathbb{V}_h$ be the finite element approximation of u defined in (2.7), that is,

$$\begin{gathered} a(u_h, v - u_h) \geqq (f, v - u_h) \quad \forall v \in \mathbb{V}_h, \\ u_h \le r_h\psi, \quad v \le r_h\psi. \end{gathered} \tag{4.1}$$

Now, let σ_h be a mapping from $L^\infty(\Omega)$ into $\mathbb{V}_h$, defined by

$$u_h = \sigma_h(\psi). \tag{4.2}$$

The mapping σ_h possesses analogous properties to those of the mapping σ (see Proposition 2.5) provided the d.m.p is satisfied.

Proposition 4.1. *The mapping σ_h is increasing, concave, and Lipschitz continuous with respect to ψ.*

4.1. The discrete system of QVIs. We define the discrete system of QVIs as follows: find $U_h = (u_h^1,\dots,u_h^J) \in (\mathbb{V}_h)^J$ such that

$$\begin{gathered} a^i(u_h^i, v - u_h^i) \geqq (f^i, v - u_h^i) \quad \forall v \in \mathbb{V}_h, \\ u_h^i \le r_h M u_h^i, \quad u_h^i \ge 0, \quad v \le r_h M u_h^i. \end{gathered} \tag{4.3}$$

4.2. Existence and uniqueness. The existence and uniqueness of a solution to system (4.3) can be shown similarly to that of the continuous case provided the d.m.p is satisfied. Indeed, the key idea for proving that consists in associating with this system the following discrete fixed point mapping:

$$\begin{gathered} T_h : \mathbb{H}^+ \longrightarrow (\mathbb{V}_h)^J, \\ W \longrightarrow T_h W = \zeta_h = (\zeta_h^1,\dots,\zeta_h^J), \end{gathered} \tag{4.4}$$

where $\zeta_h^i = \sigma_h(Mw^i)$ is the solution of the following discrete VI:

$$\begin{gathered} a^i(\zeta_h^i, v - \zeta_h^i) \geqq (f^i, v - \zeta_h^i) \quad \forall v \in \mathbb{V}_h, \\ \zeta_h^i \le r_h M w^i, \quad v \le r_h M w^i. \end{gathered} \tag{4.5}$$

Remark 4.2. Under the d.m.p, the mapping T_h possesses analogous properties to that of mapping T (see Propositions 3.1, 3.2, 3.3, 3.4, and 3.8). The proofs of such properties will not be given as they are very similar to those of the continuous case. We just list them below.

4.2.1. Some properties of the mapping T_h. Let $\overline{U}_h^0 = (\overline{u}_h^{1,0},\dots,\overline{u}_h^{J,0})$ be the discrete analogue to $\overline{U}^0$ defined in (3.4):

$$a^i(\overline{u}_h^{i,0}, v) = (f^i, v) \quad \forall v \in \mathbb{V}_h. \tag{4.6}$$

Then, we have the discrete analogues to Propositions 2.6, 3.1, 3.2, and 3.3, respectively.

Proposition 4.3. *Let $\mathbb{C}_h = \{W \in (L^\infty(\Omega))^J$ such that $0 \le W \le \overline{U}_h^0\}$. Then T_h maps $\mathbb{C}_h$ into itself.*

Proposition 4.4. *The mapping T_h is increasing and concave on $\mathbb{H}^+$.*

Proposition 4.5. *The mapping T_h is Lipschitz continuous on $\mathbb{H}^+$, that is,*

$$\|T_h W - T_h \tilde{W}\|_\infty \le \|W - \tilde{W}\|_\infty \quad \forall W, \tilde{W} \in \mathbb{H}^+. \tag{4.7}$$

Remark 4.6. It is not hard to see that the solution of system of QVIs (4.3) is a fixed point of T_h, that is, $U_h = T_h U_h$. Therefore, as in the continuous problem, one can associate with T_h the following iterative scheme.

4.2.2. A discrete iterative scheme of Bensoussan-Lions type. Starting from $\overline{U}_h^0$ solution of (4.6) (resp., from $\underline{U}_h^0 = (0,\dots,0)$), we define

$$\overline{U}_h^{n+1} = T_h \overline{U}_h^n \quad n = 0,1,\dots, \tag{4.8}$$

respectively

$$\underline{U}_h^{n+1} = T_h \underline{U}_h^n \quad n = 0,1,\dots. \tag{4.9}$$

Then, by analogy with the continuous problem, using the following intermediate results, we are able to prove the convergence of the discrete iterative scheme to the solution of system (4.3).

Lemma 4.7. *Let $0 < \lambda < \inf(k/\|\overline{U}_h^0\|_\infty, 1)$. Then, under the d.m.p, $T_h(0) \geqq \lambda \cdot \overline{U}_h^0$.*

Proposition 4.8. *Let $\gamma \in]0,1]$ and $W, \tilde{W} \in \mathbb{C}$ such that*

$$W - \tilde{W} \le \gamma W. \tag{4.10}$$

Then

$$T_h W - T_h \tilde{W} \le \gamma(1-\lambda) T_h W. \tag{4.11}$$

Theorem 4.9. *Under the d.m.p and the conditions of Propositions 4.3, 4.4, and 4.8, the sequences $(\overline{U}_h^n)$ and $(\underline{U}_h^n)$ are monotone and well defined in $\mathbb{C}_h$. Moreover, they converge, respectively, from above and below to the unique solution of system (4.3).*

Proof. Very similar to that of Theorem 3.9. □

4.2.3. Rate of convergence of the discrete iterative scheme

Proposition 4.10. *Under the d.m.p, the discrete analogues to estimates (3.24) and (3.25) hold*

$$||\overline{U}_h^n - U_h||_\infty \le (1-\lambda)^n ||\overline{U}_h^0||_\infty, \tag{4.12}$$

$$||\underline{U}_h^n - U_h||_\infty \le (1-\lambda)^n ||\overline{U}_h^0||_\infty. \tag{4.13}$$

Proof. It is exactly the same as that of Proposition 3.11. □

5. The finite element error analysis. This section is devoted to demonstrate that the proposed method is quasi-optimally accurate in $L^\infty(\Omega)$. For this purpose, we need first to introduce an auxiliary sequence of discrete VIs and next prove a fundamental lemma.

From now on, C will denote a constant independent of both h and n.

5.1. An auxiliary sequence of discrete VIs. Let $\overline{U}^n = (\overline{u}^{1,n},\dots,\overline{u}^{n,J})$ be the sequence defined in (3.19). We then introduce the following discrete sequence:

$$\tilde{U}_h^{n+1} = T_h\overline{U}^n, \quad n = 0,1,\dots, \text{ with } \tilde{U}_h^0 = \overline{U}_h^0, \tag{5.1}$$

where $\overline{U}_h^0$ is defined in (4.6) and for any $n \geqq 1$, $\tilde{u}_h^{i,n}$ is solution to the following discrete VI:

$$\begin{gathered} a^i(\tilde{u}_h^{i,n+1}, v - \tilde{u}_h^{i,n+1}) \geqq (f^i, v - \tilde{u}_h^{i,n+1}) \quad \forall v \in \mathbb{V}_h, \\ \tilde{u}_h^{i,n+1} \le r_h M\overline{u}^{i,n}, \quad v \le r_h M\overline{u}^{i,n}. \end{gathered} \tag{5.2}$$

We notice that $\tilde{u}_h^{i,n}$, solution of (5.2), represents the piecewise finite element approximation of $\overline{u}^{i,n}$, the ith component of $\overline{U}^n$. Therefore, using the regularity result provided by Lemma 5.1 and next adapting [11], we have the optimal uniform error estimate given below.

Lemma 5.1. *For any $i = 1,\dots,J$,*

$$\max_{n\ge 0}\left(||\overline{u}^{i,n}||_{W^{2,p}(\Omega)}, ||\underline{u}^{i,n}||_{W^{2,p}(\Omega)}\right) \le C, \quad 2 \le p < \infty, \tag{5.3}$$

where C is a constant independent of n.

Proof. We know that $\overline{u}^{i,1} = \sigma(M\overline{u}^{i,0})$ is a solution to the VI with obstacle $\psi = k + \inf u^{\mu,0}$, $\mu \ne i$ and $\overline{u}^{i,0} \in W^{2,p}(\Omega)$. So, $\|\psi\|_{W^{1,\infty}(\Omega)} \le C_1$ and, therefore, as in [3, Lemma 2.3, page 372], we get $\mathcal{A}^i\psi \ge -c_1$ in the sense of $H^{-1}(\Omega)$. Hence, by Lemma 2.2 and Theorem 2.3, it follows that $\|\overline{u}^{i,1}\|_{W^{2,p}(\Omega)} \le C_2$.

Now, assume that $\|\overline{u}^{i,n-1}\|_{W^{2,p}(\Omega)} \le C_3$ with C_3 independent of n. Then, $\psi = k + \inf u^{\mu,n-1}$ satisfies $\|\psi\|_{W^{1,\infty}(\Omega)} \le C_4$, $\mu \ne i$. So, using the same arguments as before, we get $\mathcal{A}^i\psi \ge -c_2$ in the sense of $H^{-1}(\Omega)$ with c independent of n, and therefore $\|\overline{u}^{i,n}\|_{W^{2,p}(\Omega)} \le C$, where C is a constant independent of n.

(The proof of $\|\underline{u}^{i,n}\|_{W^{2,p}(\Omega)} \le C$ is exactly as above.) □

Theorem 5.2. *Under the conditions of Lemma 5.1,*

$$||\overline{U}^n - \tilde{U}_h^n||_\infty \le Ch^2|\log h|^2. \tag{5.4}$$

The following lemma plays a crucial role in proving the main result.

Lemma 5.3. *Let* $(\overline{U}^n)$, $(\overline{U}_h^n)$, *and* $(\tilde{U}_h^n)$ *be the sequences defined in (3.19), (4.8), and (5.1), respectively. Then*

$$||\overline{U}^n - \overline{U}_h^n||_\infty \le \sum_{p=0}^{n} ||\overline{U}^p - \tilde{U}_h^p||_\infty. \tag{5.5}$$

Proof. We prove this lemma by induction. Indeed, estimation (5.5) is true for $n = 0$ since $\tilde{U}_h^0 = \overline{U}_h^0$. Also, knowing that

$$\overline{U}^1 = T\overline{U}^0, \qquad \overline{U}_h^1 = T_h\overline{U}_h^0, \qquad \tilde{U}_h^1 = T_h\overline{U}^0, \tag{5.6}$$

it follows that

$$\begin{aligned} ||\overline{U}^1 - \overline{U}_h^1||_\infty &\le ||\overline{U}^1 - \tilde{U}_h^1||_\infty + ||\tilde{U}_h^1 - \overline{U}_h^1||_\infty \\ &\le ||\overline{U} - \tilde{U}_h^1||_\infty + ||T_h\overline{U}^0 - T_h\overline{U}_h^0||_\infty. \end{aligned} \tag{5.7}$$

So, thanks to the Lipschitz continuity property of T_h, we get

$$\begin{aligned} ||\overline{U}^1 - \overline{U}_h^1||_\infty &\le ||\overline{U}^1 - \tilde{U}_h^1||_\infty + ||\overline{U}^0 - \overline{U}_h^0||_\infty \\ &\le \sum_{p=0}^{1} ||\overline{U}^p - \tilde{U}_h^p||_\infty. \end{aligned} \tag{5.8}$$

Now, assume that

$$||\overline{U}^{n-1} - \overline{U}_h^{n-1}||_\infty \le \sum_{p=0}^{n-1} ||\overline{U}^p - \tilde{U}_h^p||_\infty. \tag{5.9}$$

Then

$$\begin{aligned} ||\overline{U}^n - \overline{U}_h^n||_\infty &\le ||\overline{U}^n - \tilde{U}_h^n||_\infty + ||\tilde{U}_h^n - \overline{U}_h^n||_\infty \\ &\le ||\overline{U}^n - \tilde{U}_h^n||_\infty + ||T_h\overline{U}^{n-1} - T_h\overline{U}_h^{n-1}||_\infty. \end{aligned} \tag{5.10}$$

Using again the Lipschitz continuity of T_h, it follows that (5.10) is less than or equal to

$$\begin{aligned} ||\overline{U}^n - \tilde{U}_h^n||_\infty + ||\tilde{U}_h^{n-1} - \overline{U}_h^{n-1}||_\infty &\le ||\overline{U}^n - \tilde{U}_h^n||_\infty + \sum_{p=0}^{n-1} ||\overline{U}^p - \tilde{U}_h^p||_\infty \\ &\le \sum_{p=0}^{n} ||\overline{U}^p - \tilde{U}_h^p||_\infty, \end{aligned} \tag{5.11}$$

which completes the proof of Lemma 5.3. □

Now, guided by Lemma 5.3, Propositions 3.11 and 4.10, and Theorem 5.2, we are in a position to demonstrate our main result.

5.2. L^∞-error estimate for the system of QVIs (1.1)

Theorem 5.4.

$$||U - U_h||_\infty \le Ch^2 |\log h|^3, \tag{5.12}$$

$$||U - U_h||_{1,\infty} \le Ch |\log h|^3, \tag{5.13}$$

where

$$\|U\|_{1,\infty} = \max_{1 \le i \le J} ||u^i||_{W^{1,\infty}(\Omega)}. \tag{5.14}$$

Proof. Using estimates (3.24), (4.12), (5.4), and (5.5), we have

$$\begin{aligned} ||U - U_h||_\infty &\le ||U - \overline{U}^n||_\infty + ||\overline{U}^n - \overline{U}_h^n|| + ||\overline{U}_h^n - U_h||_\infty \\ &\le ||U - \overline{U}^n||_\infty + \sum_{p=0}^{n} ||\overline{U}^p - \tilde{U}_h^p||_\infty + ||\overline{U}_h^n - U_h||_\infty \\ &\le ||U^0 - \overline{U}_h^0||_\infty + \sum_{p=1}^{n} ||\overline{U}^p - \tilde{U}_h^p||_\infty + ||U - \overline{U}^n||_\infty + ||\overline{U}_h^n - U_h||_\infty \\ &\le Ch^2 |\log h|^{3/2} + n \cdot Ch^2 |\log h|^2 + (1-\lambda)^n ||\overline{U}^0||_\infty + (1-\lambda)^n ||\overline{U}_h^0||_\infty, \end{aligned} \tag{5.15}$$

where we have also used the standard uniform error estimate

$$||U^0 - \overline{U}_h^0||_\infty \le Ch^2 |\log h|^{3/2} \tag{5.16}$$

(cf. [8, 14]). Finally, letting $(1-\lambda)^n = h^2$, we get the desired result.

The $W^{1,\infty}$-error estimate (5.13) follows immediately from standard inverse inequality (cf. [8]). □

Conclusion. (1) We have established a convergence order in the L^∞-norm for a coercive system of QVIs. A future paper will be devoted to the noncoercive case for which a different approach will be developed and analyzed.

(2) It is also important to notice that the error estimate obtained in this paper contains an extra power in $\log h$ than expected. We believe that this is due to the approach followed.

(3) The same approach may also be extended to other important problems such as the system of QVIs related to games theory [3].

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