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A MIN-MAX THEOREM AND ITS APPLICATIONS TO NONCONSERVATIVE SYSTEMS

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A nonvariational generation of a min-max principle by A. Lazer is made. And it is used to prove a new existence results for a nonconservative systems of ordinary differential equations with resonance.

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1. Introduction and lemmas. Let X and Y be two subspaces of a real Hilbert space H such that $H = X \oplus Y$. Let $f : H \to \mathbb{R}$ be of class C^2 and denote by ∇f and $\nabla^2 f$ the gradient and the Hessian of f, respectively. In 1975, Lazer et al. [3] under the following conditions:

$$\begin{aligned} &\langle \nabla^2 f(u)h, h\rangle \le -m_1\|h\|^2, \quad m_1 > 0,\ \forall h \in X,\ \forall u \in H; \\ &\langle \nabla^2 f(u)k, k\rangle \ge m_2\|k\|^2, \quad m_2 > 0,\ \forall k \in Y,\ \forall u \in H \end{aligned} \tag{1.1}$$

proved that f has a unique critical point, that is, there exists a unique $v_0 \in H$ such that $\nabla f(v_0) = 0$. Moreover, this critical point is characterized by the equality

$$f(v_0) = \max_{x\in X}\min_{y\in Y} f(x+y). \tag{1.2}$$

In [5], with the following conditions:

$$\begin{aligned} &\langle \nabla^2 f(x+y)h, h\rangle \le -\beta(\|x\|)\|h\|^2, \quad \int_1^\infty \beta(s)ds = \infty, \quad \forall x, h \in X,\ \forall y \in Y; \\ &\langle \nabla^2 f(x+y)k, k\rangle \ge \alpha(\|y\|)\|k\|^2, \quad \int_1^\infty \alpha(s)ds = \infty, \quad \forall x \in X,\ \forall y, k \in Y, \end{aligned} \tag{1.3}$$

where $\alpha(s)$ and $\beta(s)$ are two continuous nonincreasing functions from $[0,\infty)$ to $(0,\infty)$, it is proved that f has a unique critical point v_0 such that $f(v_0) = \max_{x\in X}\min_{y\in Y} f(x+y)$. These results were generalized in [6] and especially for a nonselfadjoint extension of the results of Lazer. This extension was applied in [6] to prove that if the following conditions hold:

$$N^2 < \gamma_1 \le \gamma_2 < (N+1)^2, \quad \gamma_1 I \le \nabla^2 G(u) \le \gamma_2 I, \tag{1.4}$$

where N is a nonnegative integer and I is an $n\times n$ matrix, then the following differential equations system has a unique 2π-periodic solution:

$$u''(t)+Au'(t)+\nabla G(u)=e(t), \tag{1.5}$$

where A is a constant symmetric matrix. System (1.5) is included in the following nonconservative system (1.6), and assume the following:

$$u''(t)+Au'(t)+\nabla G(u,t)=e(t). \tag{1.6}$$

With the use of a nonvariational version of a max-min principle inspired by [5, 6], in Section 2 we generalize these unique existence results of system (1.6) to a more general case. To be more precise, we apply a min-max lemma to the periodic boundary value problem of the nonconservative system (1.6) and assume that the following conditions hold:

$$\begin{gathered} B_1+\alpha(\|u\|)I\le\nabla^2G(u,t)\le B_2-\beta(\|u\|)I, \\ \int_1^{+\infty}\min\{\alpha(s),\beta(s)\}ds=+\infty, \end{gathered} \tag{1.7}$$

where $u\in R^n$, B_1 and B_2 are two real symmetric matrices, and the eigenvalues of B_1 and B_2 are N_i^2 and $(N_i+1)^2$, $i=1,\dots,n$, respectively; here, N_i, $i=1,\dots,n$ are nonnegative integers and $\alpha(s)$ and $\beta(s)$ are two positive nonincreasing functions for $s\in[0,\infty)$.

In Section 3, we show with some examples that our main results extend the results known so far.

We firstly employ the following lemma from [9].

Lemma 1.1 (see [9]). *Assume that H is a Hilbert space. Let $T\in C^1(H,H)$, $T'(u)\in\operatorname{Isom}(H;H)$, for all $u\in H$. Then, T is a global diffeomorphism onto H if there exists a continuous map $\omega:R_+\to R_+\setminus\{0\}$ such that*

$$\int_1^{+\infty}\frac{ds}{\omega(s)}=+\infty,\qquad \|T'(u)^{-1}\|\le\omega(\|u\|). \tag{1.8}$$

With this lemma, we can prove the following lemma.

Lemma 1.2 (see [4]). *Let X and Y be two closed subspaces of a real Hilbert space H, and $H=X\oplus Y$. Suppose that $T:H\to H$ is a C^1-mapping. If there exist two continuous functions $\alpha:[0,\infty)\to(0,\infty)$ and $\beta:[0,\infty)\to(0,\infty)$ such that*

$$\begin{gathered} \langle T'(u)x,x\rangle\le-\alpha(\|u\|)\|x\|^2, \\ \langle T'(u)y,y\rangle\ge\beta(\|u\|)\|y\|^2, \end{gathered} \tag{1.9}$$

$$\langle T'(u)x,y\rangle=\langle x,T'(u)y\rangle \tag{1.10}$$

for arbitrary $u \in H$, $x \in X$, $y \in Y$, *and*

$$\int_1^{+\infty} \min\{\alpha(s), \beta(s)\} ds = +\infty, \tag{1.11}$$

then T *is a diffeomorphism from* H *onto* H.

The following lemma is required in the proof of Theorem 2.2.

Lemma 1.3 (see [2]). *Let* H *be a vector space such that for subspaces* Y *and* Z, $H = Z \oplus Y$. *If* Z *is finite dimensional and* X *is a subspace of* H *such that* $X \cap Y = \{0\}$ *and* dimension X = dimension Z, *then* $H = X \oplus Y$.

2. Unique existence.

Assume that $G(u,t)$ is continuous for $(u,t) \in R^n \times [0,2\pi]$ and twice continuously differiable about u. Denote by $\nabla G(u,t)$ and $\nabla^2 G(u,t)$ the gradient and the Hessian of $G(u,t)$, respectively. We will investigate the unique existence of periodic solutions for system (1.6).

Firstly, we introduce the following definition.

Definition 2.1. The real symmetric matrix A is called admissible with two real symmetric matrices B_1 and B_2 if there exist orthogonal matrices P_1 and P_2 such that $P_1^T B_1 P_1$, $P_2^T B_2 P_2$, and $P_1^T A P_2$ are simultaneously diagonal matrices.

Theorem 2.2. *If conditions (1.7) hold for all* $t \in [0,2\pi]$, *all* $u \in R^n$, *and for* A *is admissible with the matrices* B_1 *and* B_2, *then there exists a unique* 2π*-periodic solution to system (1.6).*

Proof. Because A is admissible with the matrices B_1 and B_2, and conditions (1.7) hold for all $t \in [0,2\pi]$ and all $u \in R^n$, we can get orthogonal matrices $P_1 = (a_1, a_2, \dots, a_n)$, $P_2 = (b_1, b_2, \dots, b_n)$, $P_1^T B_1 P_1 = \operatorname{diag}(N_1^2, \dots, N_n^2)$, $P_2^T B_2 P_2 = \operatorname{diag}((N_1+1)^2, \dots, (N_n+1)^2)$, and $P_1^T A P_2 = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$. Clearly, a_i and b_i are the eigenvectors of B_1 and B_2, respectively, corresponding to the eigenvalues N_i^2 and $(N_i+1)^2$, which satisfy

$$a_i^T a_j = b_i^T b_j = \delta_{ij}, \quad i,j = 1,2,\dots,n, \tag{2.1}$$

where $\delta_{ij} = 0$, $i \neq j$; $\delta_{ij} = 1$, $i = j$. Define

$$\begin{aligned} V = \{ & v(t) = (v_1(t), \dots, v_n(t))^T \mid v_i(0) = v_i(2\pi),\ i = 0,1,\dots,n; \\ & v(t) \text{ absolutely continuous and } v'(t) \in L^2[0,2\pi]\}, \end{aligned} \tag{2.2}$$

and it is easy to see that V is a Hilbert space with the following inner product:

$$\langle u, v \rangle = \int_0^{2\pi} [u'^T(t) v'(t) + u^T(t) v(t)] dt. \tag{2.3}$$

Denote by $\|\cdot\|_V$ the norm induced by this inner product, and define subspaces of V as follows:

$$\begin{aligned} X &= \left\{ x(t) = \sum_{i=1}^{n} f_i(t) a_i \mid f_i(t) = c_{i0} + \sum_{m=1}^{N_i} (c_{im} \cos mt + d_{im} \sin mt) \right\}; \\ Y &= \left\{ y(t) = \sum_{i=1}^{n} g_i(t) b_i \mid g_i(t) = \sum_{m=N_i+1}^{\infty} (p_{im} \cos mt + q_{im} \sin mt) \right\}; \\ Z &= \left\{ z(t) = \sum_{i=1}^{n} h_i(t) b_i \mid h_i(t) = p_{i0} + \sum_{m=1}^{N_i} (p_{im} \cos mt + q_{im} \sin mt) \right\}, \end{aligned} \tag{2.4}$$

where N_i, $i = 1, \dots, n$ are as in (1.7) and c_{im}, d_{im}, p_{im}, and q_{im} are constants. Obviously, $V = Z \oplus Y$. Using the Riesz representation theorem, define a mapping $T : V \to V$ by

$$\langle T(u), v \rangle = \int_0^{2\pi} [u'^T(t) v'(t) - v^T(t) A u'(t) - v^T(t) \nabla G(u(t), t)] dt \tag{2.5}$$

for arbitrary $v \in V$. We observe that T is defined implicitly. From (2.5) and the fact that G is C^2, it can be proved that T is a C^1-mapping and that

$$\langle T'(u) w, v \rangle = \int_0^{2\pi} [w'^T v'(t) - v^T(t) A w'^T(t) - v^T(t) \nabla^2 G(u, t) w(t)] dt \tag{2.6}$$

for all $v(t), u(t), w(t) \in V$. Again, from the Riesz representation theorem, there exists an element $d \in V$ satisfying

$$\langle d, v \rangle = -\int_0^{2\pi} v^T(t) e(t) dt. \tag{2.7}$$

It can be proved that u is a 2π-periodic solution to (1.6) if and only if u satisfies the operator equation

$$T(u) = d. \tag{2.8}$$

We will next show that T satisfies the conditions of Lemma 1.2. This will, in turn, imply that (1.6) has a unique 2π-periodic solution. For any $x \in X$ and $u \in V$, we have that

$$\langle T'(u) x, x \rangle = \int_0^{2\pi} [x'^T(t) x'(t) - x^T(t) A x'(t) - x^T(t) \nabla^2 G(u, t) x(t)] dt, \tag{2.9}$$

where

$$\begin{aligned} \int_0^{2\pi} x'^T(t) x'(t) dt &= \int_0^{2\pi} \sum_{i=1}^{n} f_i'^2(t) dt \le \sum_{i=1}^{n} N_i^2 \int_0^{2\pi} f_i^2(t) dt; \\ \int_0^{2\pi} x^T(t) A x'(t) dt &= \frac{1}{2} x^T(t) A x(t) |_0^{2\pi} = 0. \end{aligned} \tag{2.10}$$

By (1.7), we have

$$\begin{aligned}
&\int_0^{2\pi} x^T(t)\nabla^2 G(u,t)x(t)dt \\
&\quad \ge \int_0^{2\pi} x^T(t)B_1x(t)dt+\alpha(\|u\|_V)\int_0^{2\pi} x^T(t)x(t)dt \\
&\quad = \int_0^{2\pi}\sum_{i=1}^n\sum_{j=1}^n f_i(t)f_j(t)a_i^TB_1a_j\,dt+\alpha(\|u\|_V)\int_0^{2\pi} x^T(t)x(t)dt \\
&\quad = \sum_{i=1}^n N_i^2\int_0^{2\pi} f_i^2(t)dt+\alpha(\|u\|_V)\int_0^{2\pi} x^T(t)x(t)dt; \\
&\|x\|_V^2 = \int_0^{2\pi} x^T(t)x(t)dt+\int_0^{2\pi} x'^T(t)x'(t)dt \\
&\quad \le (M^2+1)\int_0^{2\pi} x^T(t)x(t)dt,
\end{aligned} \tag{2.11}$$

where $M=\max_{1\le i\le n}\{N_i\}$, therefore

$$\langle T'(u)x,x\rangle \le -\frac{\alpha(\|u\|_V)}{M^2+1}\|x\|_V^2. \tag{2.12}$$

Similarly, from

$$\begin{aligned}
\int_0^{2\pi} y'^T(t)y'(t)dt &\ge \sum_{i=1}^n (N_i+1)^2\int_0^{2\pi} g_i^2(t)dt, \\
-\int_0^{2\pi} y^T(t)\nabla^2G(u,t)y(t)dt &\ge -\int_0^{2\pi} y^T(t)B_2y(t)dt \\
&\quad +\beta(\|u\|_V)\int_0^{2\pi} y^T(t)y(t)dt,
\end{aligned} \tag{2.13}$$

we can get that for all $y\in Y$ and all $u\in V$,

$$\begin{aligned}
&\int_0^{2\pi}\{[1+(M+1)^2][y'^T(t)y'(t)-y^T(t)\nabla^2G(u,t)y(t)] \\
&\qquad -\beta(\|u\|_V)[y'^Ty'(t)+y^T(t)y(t)]\}dt \\
&\quad = [1+(M+1)^2-\beta(\|u\|_V)]\int_0^{2\pi} y'^T(t)y'(t)dt \\
&\qquad -\int_0^{2\pi}[1+(M+1)^2]y^T(t)\nabla^2G(u,t)y(t)dt-\beta(\|u\|_V)\int_0^{2\pi} y^T(t)y(t)dt \\
&\quad \ge [1+(M+1)^2-\beta(\|u\|_V)]\sum_{i=1}^n (N_i+1)^2\int_0^{2\pi} g_i^2(t)dt \\
&\qquad -\int_0^{2\pi}[1+(M+1)^2]y^T(t)B_2y(t)dt+(M+1)^2\beta(\|u\|_V)\int_0^{2\pi} y^T(t)y(t)dt
\end{aligned}$$

$$
\begin{aligned}
&= \left[1+(M+1)^2-\beta(\|u\|_V)\right]\sum_{i=1}^{n}(N_i+1)^2\int_0^{2\pi} g_i^2(t)\,dt \\
&\quad -\left[1+(M+1)^2\right]\sum_{i=1}^{n}(N_i+1)^2\int_0^{2\pi} g_i^2(t)\,dt+(M+1)^2\beta(\|u\|_V)\sum_{i=1}^{n}\int_0^{2\pi} g_i^2(t)\,dt \\
&= \beta(\|u\|_V)\sum_{i=1}^{n}\left[(M+1)^2-(N_i+1)^2\right]\int_0^{2\pi} g_i^2(t)\,dt \ge 0,
\end{aligned}
\tag{2.14}
$$

and from

$$
\int_0^{2\pi} y^T(t)Ay'(t)\,dt = \frac{1}{2}y^T(t)Ay(t)\big|_0^{2\pi} = 0, \tag{2.15}
$$

we can prove that for all $y \in Y$ and all $u \in V$,

$$
\begin{aligned}
\langle T'(u)y, y\rangle &= \int_0^{2\pi}\left[y'^T(t)y'(t)-y^T(t)Ay'(t)-y^T(t)\nabla^2 G(u,t)y(t)\right]dt \\
&\ge \frac{\beta(\|u\|_V)}{(M+1)^2+1}\|y\|_V^2.
\end{aligned}
\tag{2.16}
$$

Obviously, for all $x \in X$ and all $y \in Y$, we have the following:

$$
\begin{aligned}
&\langle T'(u)x, y\rangle - \langle x, T'(u)y\rangle \\
&\quad = \int_0^{2\pi}\left[x^T(t)Ay'(t)-y^T(t)Ax'(t)\right]dt \\
&\quad = \int_0^{2\pi} 2\left(f_1(t),\dots,f_n(t)\right)P_1^T A P_2\left(g_1(t),\dots,g_n(t)\right)^T dt \\
&\quad = \int_0^{2\pi}\sum_{i=1}^{n} 2\gamma_i f_i(t) g_i'(t)\,dt = 0.
\end{aligned}
\tag{2.17}
$$

Let $\alpha_1 = \alpha(s)/(M^2+1)$ and $\beta_1(s) = \beta(s)/((M+1)^2+1)$, then

$$
c(s) = \min\{\alpha_1(s), \beta_1(s)\} \ge \min\{\alpha(s), \beta(s)\}/\left((M+1)^2+1\right). \tag{2.18}
$$

Based on conditions (1.7), $\int_1^{+\infty} c(s)\,ds = +\infty$. Since $T'(u)$ is positive definite on Y and negative definite on X, we see that $X \cap Y = \{0\}$. Moreover, it is readily seen that

$$
\text{dimension}\, X = \text{dimension}\, Z = \sum_{i=1}^{n}(2N_i+1). \tag{2.19}
$$

Thus, since it was shown above that $V = Z \oplus Y$, it follows, by application of Lemma 1.3, that $V = X \oplus Y$. We may, therefore, apply Lemma 1.2 to get the conclusion of the theorem.

If we set $V = \{v(t) = (v_1(t),\dots,v_n(t))^T \mid v_i(0) = v_i(\pi) = 0,\ i = 1,\dots,n;\ v(t)$ to be absolutely continuous and $v'(t) \in L^2[0,\pi]\}$, it is easy to know that V is a Hilbert space about the following inner product:

$$\langle u,v\rangle = \int_0^{\pi} [u'^T(t)v'(t)+u^T(t)v(t)]dt. \tag{2.20}$$

Again, define the norm induced by this inner product and subspaces X, Y, and Z, correspondingly; we can prove the following theorem similarly. □

Theorem 2.3. *Assume that $G(u,t)$ is continuous and C^2-mapping with respect to u and that conditions (1.7) hold for all $t \in [0,\pi]$, all $u \in R^n$, and for A is admissible with matrices B_1 and B_2. Let $e(t)$ be a continuous function. Then, there exists a unique solution to (1.6), which satisfies boundary value condition $u(0) = u(\pi) = 0$.*

Especially, when $B_1 = N^2 I$ and $B_2 = (N+1)^2 I$, where N is natrual and I is $n\times n$ identity matrix, A is admissible with B_1 and B_2 as long as A is real symmetric. So, we have the following corollary.

Corollary 2.4. *Assume that A is real symmetric and there exist two positive continuous functions δ_1 and $\delta_2 : R^n \to \mathbb{R}$ such that for all $u \in R^n$ and all $t \in [0,2\pi]$,*

$$N^2 I < \delta_1(u)I \le \nabla^2 G(u,t) \le \delta_2(u)I < (N+1)^2. \tag{2.21}$$

Let $\rho(r) = \min\{1-\max_{\sum_{i=1}^n|\xi_i|\le r}(\delta_2(\xi)/(N+1)^2), \max_{\sum_{i=1}^n|\xi_i|\le r}(\delta_1(\xi)/N^2)-1\}$, and if $\int_1^{+\infty}\rho(r)dr = +\infty$, then the system (1.6) has a unique 2π-periodic solution.

If we set $A = 0$, system (1.6) becomes a conservative system and admissibility is trivial. So, the main conclusion in [7] (the method there is different from ours) is a corollary of Theorem 2.2.

Corollary 2.5. *Assume that there exist integers $N_i \ge 0$ such that for all $u \in R^n$ and all $t \in [0,2\pi]$,*

$$\begin{gathered} N_i^2 < \lambda_i(u,t) < (N_i+1)^2, \quad i = 1,\dots,n; \\ \delta(\|u\|,t) = \max_{\|v\|\le\|u\|}\left\{\min_{1\le i\le n}\left\{\lambda_i(u,t)-N_i^2, (N_i+1)^2-\lambda_i(u,t)\right\}\right\}, \end{gathered} \tag{2.22}$$

where $\lambda_i(u,t)$, $i = 1,2,\dots,n$ denote the eigenvalues of $\nabla^2 G(u)$. If $\int_1^{+\infty}\delta(s,t)ds = +\infty$ for all $t \in [0,2\pi]$, then there exists a 2π-periodic solution to (1.5).

Let $G(u,t) = G(u)$ and

$$c(s) = \min\{\alpha(s),\beta(s)\} \ge c_0 > 0; \tag{2.23}$$

we can get the following unique existence corollary.

Corollary 2.6. *If the real symmetric matrix A is admissible with real symmetric matrices B_1 and B_2, then assume that*

$$B_1 \le \nabla^2 G(u) \le B_2, \qquad N_i^2 < \lambda_i \le \mu_i < (N_i+1)^2, \tag{2.24}$$

where λ_i and μ_i are eigenvalues of B_1 and B_2, respectively, and N_i, $i = 1,\dots,n$ are nonnegative integers; there exists a unique 2π-periodic solution to system (1.5).

3. Examples. It should be pointed out that conditions (1.9) and (1.11) are not completely the same as (1.3). In fact, from (1.3), we know that $\alpha(\|x\|)$ depends on subspace X and $\beta(\|y\|)$ depends upon subspace Y. So, condition (1.3) is more strict than conditions (1.9). But, from (1.11), we can deduce the following conditions:

$$\int_1^{+\infty} \alpha(s)ds = +\infty, \qquad \int_1^{+\infty} \beta(s)ds = +\infty. \tag{3.1}$$

Conversely, note that (3.1) does not imply (1.11). Now, we give an example to illustrate it.

Example 3.1. First of all, we define two nondecreasing functions as follows:

$$\begin{aligned} \alpha(x) &= 2^{-(2i+1)^2}, \quad x_{2i-1} \le x < x_{2i+1}; \\ \beta(x) &= 2^{-(2i+2)^2}, \quad x_{2i} \le x < x_{2i+2}, \end{aligned} \tag{3.2}$$

where $x_i = \sum_{k=0}^{i} 2^{k^2}$, $i = 0,1,\dots$, $x_{-1} = 0$, and $\beta(x) = 1$, when $0 \le x < 1$.

It is easy to see that $\alpha(x)$ and $\beta(s)$ are two nondecreasing positive functions for all $x \in [0,+\infty)$; and the number of noncontinuous points is countable infinite. We also have

$$\begin{aligned} \int_1^{+\infty} \alpha(x)dx &= 1 + \sum_{i=1}^{+\infty} \frac{x_{2i+1} - x_{2i-1}}{2^{(2i+1)^2}} \\ &= 1 + \sum_{i=1}^{+\infty} \frac{2^{(2i+1)^2} + 2^{(2i)^2}}{2^{(2i+1)^2}} = +\infty, \\ \int_1^{+\infty} \beta(x)dx &= \sum_{i=1}^{+\infty} \frac{x_{2i} - x_{2i-2}}{2^{(2i)^2}} = \sum_{i=1}^{+\infty} \frac{2^{(2i)^2} + 2^{(2i-1)^2}}{2^{(2i)^2}} = +\infty, \\ \int_1^{+\infty} \min\{\alpha(x),\beta(x)\}dx &\le \sum_{i=0}^{+\infty} \frac{x_{2i+1} - x_{2i}}{2^{(2i+2)^2}} + \sum_{i=0}^{+\infty} \frac{x_{2i} - x_{2i-1}}{2^{(2i+1)^2}} \\ &\le \sum_{i=0}^{+\infty} \frac{2^{(2i+1)^2}}{2^{(2i+2)^2}} + \sum_{i=0}^{+\infty} \frac{2^{(2i)^2}}{2^{(2i+1)^2}} < +\infty. \end{aligned} \tag{3.3}$$

Secondly, from the definition of $\alpha(x)$ and $\beta(x)$, it is easy to make them continuous and even continuously differentiable, and then they are still positive nondecreasing and satisfy (3.1), but they do not satisfy conditions (1.11).

We can state, from the following example, that Theorem 2.3 is more general than the results of [1, 2, 3, 4, 5, 6, 7, 8].

Example 3.2. Assume that $f(t)$ is continuous and 2π-periodic in (1.6). Let

$$\begin{aligned} G(u,t) = {} & \frac{5}{4}\left(1+\frac{4}{5}\sin^2 t\right)\left(u_1^2+u_2^2\right)+\frac{3}{2}\left(1+\frac{2}{3}\sin^2 t\right)u_1u_2 \\ & + u_1\ln\left(u_1+\sqrt{1+u_1^2}\right)+u_2\ln\left(u_2+\sqrt{1+u_2^2}\right) \\ & - \sqrt{1+u_1^2}-\sqrt{1+u_2^2}+C_1u_1+C_2u_2, \end{aligned} \tag{3.4}$$

then

$$\nabla^2 G(u,t) = \begin{pmatrix} \frac{5}{2}\left(1+\frac{4}{5}\sin^2 t\right)+\frac{1}{\sqrt{1+u_1^2}} & \frac{3}{2}\left(1+\frac{2}{3}\sin^2(t)\right) \\ \frac{3}{2}\left(1+\frac{2}{3}\sin^2(t)\right) & \frac{5}{2}\left(1+\frac{4}{5}\sin^2 t\right)+\frac{1}{\sqrt{1+u_2^2}} \end{pmatrix},$$

$$\begin{aligned} \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{5}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{1+u_1^2}} & 0 \\ 0 & \frac{1}{\sqrt{1+u_2^2}} \end{pmatrix} & \le \nabla^2 G(u,t) \\ & \le \begin{pmatrix} \frac{13}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{13}{2} \end{pmatrix} - \begin{pmatrix} 1-\frac{1}{\sqrt{1+u_1^2}} & 0 \\ 0 & 1-\frac{1}{\sqrt{1+u_2^2}} \end{pmatrix}. \end{aligned} \tag{3.5}$$

It is easy to see that $G(u,t)$ satisfies (1.7). Therefore, there exists a unique 2π-periodic solution to (1.6) by Theorem 2.2, but we cannot make this conclusion from [1, 2, 3, 4, 5, 6, 7, 8].

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