IJMMS 2003:14, 853–856
PII. S0161171203203227
http://ijmms.hindawi.com

A FACTORIZATION THEOREM FOR LOGHARMONIC MAPPINGS

ZAYID ABDULHADI and YUSUF ABUMUHANNA

Received 24 March 2002

We give the necessary and sufficient condition on sense-preserving logharmonic mapping in order to be factorized as the composition of analytic function followed by a univalent logharmonic mapping.

2000 Mathematics Subject Classification: 30C55, 30C62, 49Q05.

Let D be a domain of $\mathbb{C}$ and denote by $H(D)$ the linear space of all analytic functions defined on D. A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = \left(a\frac{\overline{f}}{f}\right) f_z, \tag{1}$$

where $a \in H(D)$ and $|a(z)| < 1$ for all $z \in D$. If f does not vanish on D, then f is of the form

$$f = H \cdot \overline{G}, \tag{2}$$

where H and G are locally analytic (possibly multivalued) functions on D. On the other hand, if f vanishes at z_0, but is not identically zero, then f admits the local represenation

$$f(z) = (z - z_0)^m |z - z_0|^{2\beta m} h(z)\overline{g(z)}, \tag{3}$$

where

(a) m is a nonnegative integer,
(b) $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ and therefore $\Re\beta > -1/2$,
(c) h and g are analytic in a neighbourhood of z_0.

In particular, if D is a simply connected domain, then f admits a global representation of the form (3) (see, e.g., [2]). Univalent logharmonic mappings defined on the unit disk U have been studied extensively (for details, see, e.g., [1, 2, 3, 4, 5, 6]).

In the theory of quasiconformal mappings, it is proved that for any measurable function μ with $|\mu| < 1$, the solution of Beltrami equation $f_{\bar{z}} = \mu f_z$ can be factorized in the form $f = \psi \circ F$, where F is a univalent quasiconformal mapping and ψ is an analytic function (see [8]). Moreover, for sense-preserving harmonic mappings, the answer was negative. In [7], Duren and Hengartner gave a necessary and sufficient condition on sense-preserving harmonic mapping f for the existence of such factorization. Since logharmonic mappings are preserved under precomposition with analytic functions, it is a natural question to ask whether every sense-preserving logharmonic mapping can be factorized in the form $f = F \circ \phi$ for some univalent logharmonic mapping F and some analytic function ϕ.

It is instructive to begin with two simple examples.

Example 1. Let f be the logharmonic mapping $f(z) = z^2/|1-z|^4$ defined on the unit disc U. Then f is sense-preserving in U with dilatation $a(z) = z$. We claim that f has no decomposition of the desired form in any neighborhood of the origin. Suppose on the contrary that $f = F \circ \phi$, where ϕ is analytic near the origin and F is univalent logharmonic mapping on the range of ϕ. Then F is sense-preserving because f is. Without loss of generality, we suppose that $\phi(0) = 0$. Then F has a representation $F = \zeta H(\zeta)\overline{G(\zeta)}$, where H and G are analytic and have power series expansion

$$H(\zeta) = \sum_{n=0}^{\infty} A_n \zeta^n, \qquad G(\zeta) = \sum_{n=0}^{\infty} B_n \zeta^n, \tag{4}$$

where $|A_0| = |B_0| = 1$.

Since the analytic part of $f(z)$ is $\phi(z)H(\phi(z)) = z^2/(1-z)^2$, the function ϕ must have an expansion of the form

$$\phi(z) = c_2 z^2 + c_3 z^3 + \cdots. \tag{5}$$

It follows that $G \circ \phi$ has an expansion of the form $B_0 + C_1 z^2 + C_2 z^3 + \cdots$. However, the given form of f shows that $G(\phi(z)) = 1/(1-z)^2 = 1 + 2zz + 3z^2 + \cdots$, this leads to contradiction. Hence, f has no factorization of the form $f = F \circ \phi$ of the required form in any neighborhood of the origin.

Example 2. Let $f(z) = z^2|z^2|$ be defined in the unit disc U. Now, f is sense-preserving logharmonic mapping in U since the dilatation $a(z) = 1/3$. But here f has the desired factorization $f = F \circ \phi$, with $F(\zeta) = \zeta|\zeta|$ and $\phi(z) = z^2$.

Now, we state and prove the factorization theorem.

Theorem 3. *Let f be a nonconstant logharmonic mapping defined on a domain $D \subset \mathbb{C}$ and let a be its dilatation function. Then, f can be factorized in the form $f = F \circ \phi$, for some analytic function ϕ and some univalent logharmonic mapping F if and only if*

(a) $|a(z)| \neq 1$ *on* D*;*

(b) $f(z_1) = f(z_2)$ *implies* $a(z_1) = a(z_2)$.

Under these conditions, the representation is unique up to a conformal mapping; any other represenation $f = F_1 \circ \phi_1$ has the form $F_1 = F \circ \psi^{-1}$ and $\phi_1 = \psi \circ \phi$ for some conformal mapping defined on $\phi(D)$.

Proof. Suppose that $f = F \circ \phi$, where F is a univalent logharmonic mapping and ϕ is an analytic function. Let $A(\zeta)$ be the dilatation function of F. Then simple calculations give that $f_z = F_w(\phi)\phi'$, $f_{\overline{z}} = F_{\overline{w}}(\phi)\overline{\phi'}$, and $a(z) = A(\phi(z))$. Since F is univalent, the Jacobian is nonzero and hence $|a(z)| = |A(\phi(z))| \neq 1$ (see [2]). Also, F is univalent and $f(z_1) = f(z_2)$ implies that $\phi(z_1) = \phi(z_2)$. Hence, $a(z_1) = a(z_2)$.

Next, suppose that the two conditions are satisfied. We want to show that f can be factorized in the form $f = F(\phi)$. This is equivalent to finding a univalent continuous function G defined on $f(D)$ so that $G \circ f$ is analytic. In view of the Cauchy-Riemann conditions, this is equivalent to

$$(G_w b + G_{\overline{w}})\overline{f_z} = 0, \tag{6}$$

where $b(z) = \overline{a(z)}(f(z)/\overline{f(z)}) = f_{\overline{z}}/\overline{f_z}$.

Hence, $-b(f^{-1}(w)) = G_{\overline{w}}/G_w$. Let $\mu(w) = G_{\overline{w}}/G_w$. Now, we show that μ is well defined. Suppose that $f(z_1) = f(z_2) = w$. Then, as $b(z_1) = \overline{a(z_1)}(f(z_1)/\overline{f(z_1)})$, $b(z_2) = \overline{a(z_2)}(f(z_2)/\overline{f(z_2)})$, and $a(z_1) = a(z_2)$, it follows that $b(z_1) = b(z_2)$. Hence, $\mu(w)$ is well defined and $|\mu(w)| \neq 1$ for all $w \in f(D)$.

Let $\{D_n\}$ be an exhaustion of D, $\Omega_n = f(D_n)$ and let μ_n be the restriction of μ to Ω_n. Extend μ_n to $\overline{\mathbb{C}}$ by assuming that $\mu_n \equiv 0$ on $\mathbb{C} \setminus \Omega_n$. Then the Beltrami equation $G_{\overline{w}} = \mu_n G_w$ has a quasiconformal solution G_n from $\mathbb{C}$ on $\mathbb{C}$, see [8]. Let $G_n(\infty) = \infty$, then G_n is a homeomorphism on $\overline{\mathbb{C}}$. Replace the solution G_n with the solution

$$H_n(w) = \frac{G_n(w) - G_n(w_0)}{G_n(w_1) - G_n(w_0)}, \tag{7}$$

where $w_0, w_1 \in \Omega_1$ and $w_1 \neq w_0$. This is possible because f is not constant on D_1. Then, H_n is also a homeomorphic solution to the Beltrami equation, normalized to satisfy $H_n(w_0) = 0$, $H_n(w_1) = 1$, and $H_n(\infty) = \infty$. This and the fact that each H_n is K-quasiconformal mapping on Ω_j imply that H_n converges locally uniformly to a K-quasiconformal mapping H on Ω_j. It follows that H is a homeomorphism on Ω and H satisfies the equation

$$H_{\overline{w}} = \mu H_w \quad \text{on } \Omega. \tag{8}$$

Hence, $\phi = H \circ f$ is analytic in D.

Next, we show that $F = H^{-1}$ is logharmonic mapping. Note that $f = F \circ \phi$ was assumed to be logharmonic in D. Then, near any point $\zeta = \phi(z)$ where $\phi'(z) \neq 0$, we can then deduce that $F = f \circ \phi^{-1}$ is logharmonic, where ϕ^{-1} is a local inverse. But F is locally bounded, so the (isolated) images of critical points of ϕ are removable, and F is logharmonic mapping on $\phi(D)$.

Finally, we prove the uniqueness. Suppose that $f = F \circ \phi = F_0 \circ \phi_0$. If we let $G_0 = {F_0}^{-1}$, then $G_0 \circ f = \phi_0$ is nonconstant and analytic, and $G_{0\overline{w}} = \mu G_{0w}$. But the solution of this Beltrami equation is unique; hence, $G_0 = G$. This completes the proof of the theorem. □

References

[1] Z. Abdulhadi, *Close-to-starlike logharmonic mappings*, Int. J. Math. Math. Sci. **19** (1996), no. 3, 563–574.

[2] Z. Abdulhadi and D. Bshouty, *Univalent functions in* $H\cdot\overline{H}(D)$, Trans. Amer. Math. Soc. **305** (1988), no. 2, 841–849.

[3] Z. Abdulhadi and W. Hengartner, *Spirallike logharmonic mappings*, Complex Variables Theory Appl. **9** (1987), no. 2-3, 121–130.

[4] ______, *Univalent harmonic mappings on the left half-plane with periodic dilatations*, Univalent Functions, Fractional Calculus, and Their Applications (Kōriyama, 1988), Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1989, pp. 13–28.

[5] ______, *Univalent logharmonic extensions onto the unit disk or onto an annulus*, Current Topics in Analytic Function Theory, World Scientific Publishing, New Jersey, 1992, pp. 1–12.

[6] ______, *One pointed univalent logharmonic mappings*, J. Math. Anal. Appl. **203** (1996), no. 2, 333–351.

[7] P. Duren and W. Hengartner, *A decomposition theorem for planar harmonic mappings*, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1191–1195.

[8] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, 2nd ed., Springer-Verlag, New York, 1973.

Zayid Abdulhadi: Department of Computer Science, Mathematics and Statistics, American University of Sharjah (AUS), P.O. Box 26666, Sharjah, United Arab Emirates
E-mail address: zahadi@aus.ac.ae

Yusuf Abumuhanna: Department of Computer Science, Mathematics and Statistics, American University of Sharjah (AUS), P.O. Box 26666, Sharjah, United Arab Emirates
E-mail address: ymuhanna@aus.ac.ae