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AN EXTENSION OF A BEST APPROXIMATION THEOREM

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ABSTRACT. In this paper an extension of a theorem of Prolla is given and several interesting corollaries are derived. Fixed point results are also given in the end.

KEY WORDS AND PHRASES. Almost affine map, coincidence point, approximatively compact set, fixed points.
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1. INTRODUCTION.

Recently several researchers have given extensions of the following well-known theorem of Ky Fan [4] on best approximation.

THEOREM 1. Let C be a compact convex subset of a normed linear space X and $f : C \to X$ a continuous function. Then there is an $x_0 \in C$ such that

$$\|x_0 - fx_0\| = d(fx_0, C) = \inf\{\|fx_0 - y\| : y \in C\} .$$

There are several proofs of this theorem using the KKM-map principle, variational inequality and fixed point theory. The above theorem has interesting applications in fixed point theory and approximation theory.

Prolla [8] gave the following.

THEOREM 2. Let C be a nonempty compact convex subset of a normed linear space X, and $g : C \to C$ continuous, almost-affine, onto map. If $f : C \to X$ is a continuous function then there is a $y_0 \in C$ such that

$$\|gy_0 - fy_0\| = d(fy_0, C) .$$

Note. In case $g = I$, an identity function, then Theorem 1 is obtained.

The purpose of this paper is to extend Theorem 2 and derive a few interesting corollaries.

We need the following.

Let X be a Banach space and C a nonempty subset of X. Let $x \in X$ and denote $d(x, C) = \inf\{\|x - y\| : y \in C\}$. In case $d(x, C) = \|x - y\|$ for $y \in C$, then y is said to be an element of best approximation to x. The set of best approximation to x is given by

$$P(x) = \{y \in C : \|x - y\| = d(x, C)\} .$$

The map $P : X \to 2^C$ is called the metric projection onto C. If $Px \neq \emptyset$ for all $x \in X$ then C is called a proximinal set. In case $P(x)$ contains at most one element for each $x \in X$ then C is called a Chebyshev set.

A subset C of X is called an approximatively compact if for every $x \in X$ and every sequence $\{y_n\}$ in C with $\lim\limits_{n\to\infty} \|x - y_n\| = d(x, C)$ there exists a subsequence $\{y_{n_i}\}$ converging to an element of C.

A compact set is always approximatively compact but converse is not true. For example, a closed convex set in a Hilbert space is approximatively compact but not compact.

For an approximately compact set C the following holds.

i) $Px \neq \emptyset$ for each $x \in X$;

ii) C is closed;

iii) Px is compact;

iv) if C is convex then Px is convex;

v) the metric projection $P : X \to 2^C$ is upper semicontinuous (see [9] or [11]);

vi) $P(A) = \cup\{P(x) : x \in A\}$ is compact for any compact subset A in X.

Let X and Y be normed linear spaces and 2^Y denote the set of all nonempty subsets of Y. A multivalued mapping $F : X \to 2^Y$ is upper semicontinuous (usc) if $F^{-1}(A) = \{x \in X : Fx \cap A \neq \emptyset\}$ is closed in X for each closed set A in Y.

A multivalued map F is said to be compact if $F(X)$ is contained in a compact subset of Y. F is said to be acyclic if Fx is nonempty, compact and acyclic subset of Y for each $x \in X$.

A multivalued map $F : X \to X$ (X is a metric space) is said to be admissible if there are maps

$$F_i : X_i \to X_{i+1} \qquad i = 0, 1, 2, ..., n \qquad X_0 = X_{n+1} = X$$

such that:

i) $F = F_n F_{n-1} ... F_0$;

ii) F_i is acyclic and usc for each i;

iii) X_i are metric spaces for each $i = 1, 2, ..., n$ (see [7]).

The following theorem will be used [7] in our work.

THEOREM 3. Let C be a convex subset of a Banach space X and $F : C \to C$ an admissible compact map. Then F has a fixed point.

Let C be a convex subset of X. A map $g : C \to X$ is almost-affine if it satisfies

$$\|g(\lambda x_1 + (1 - \lambda)x_2) - y\| \leq \lambda\|gx_1 - y\| + (1 - \lambda)\|gx_2 - y\|$$

for all x_1, $x_2 \in C$, $y \in X$ and $0 < \lambda < 1$.

g is an affine map if

$$g(\lambda x_1 + (1 - \lambda)x_2) = \lambda g x_1 + (1 - \lambda) g x_2, \quad \lambda \in (0, 1).$$

If $g : C \to C$ is a single-valued function then g is said to be proper if $g^{-1}(A)$ is compact for A compact.

The following is the main result.

THEOREM 4. Let C be a nonempty convex subset of a normed linear space X and $P : X \to 2^C$ the metric projection satisfying

i) $P(x) = \{y \in C : \|x - y\| \leq \|x - z\| \text{ for all } z \in C\} \neq \emptyset$ for each $x \in X$ and

ii) P sends compact subsets of X onto compact subsets of C.

Let $g : C \to C$ be a continuous, onto, proper and $g^{-1}(z)$ an acyclic subset of C for every $z \in C$.

Then for every continuous map $f : C \to X$ with $f(C)$ relatively compact there exists a $y_0 \in C$ such that

$$\|gy_0 - fy_0\| = d(fy_0, C).$$

Note. 1. In case C is an approximatively compact set then conditions i) and ii) are satisfied by P.

2. In case C is a compact convex set then the condition that $f(C)$ is relatively compact is not required since the continuous image of a compact set is compact. The condition that g is proper is also not needed, since g is continuous so for any compact set D in C, $g^{-1}(D)$ is a closed subset of a compact set C and hence is compact.

The proof is on the same lines as in [3].

PROOF. Let $P : X \to 2^C$ be the metric projection. Define a multivalued map $F : C \to C$ by

$$Fx = \{y \in C : gy \in P(f(x))\}.$$

Then $Fx \neq \emptyset$ for each $x \in C$ since g is an onto map. $F(x)$ is closed and acyclic (see [3]). We show that F is upper semicontinuous. This will be done as in [10].

Let B be a closed subset of C and z a limit point of $F^{-1}(B)$. We choose $\{z_n\} \subseteq F^{-1}(B) \subseteq C$ such that $z_n \to z$. We show that $z \in F^{-1}(B)$. Since for each n, $F(z_n) \cap B \neq \emptyset$ we have $\{y_n\} \subseteq C$ with $y_n \subseteq F(z_n) \cap B$.

Then for each n,

$$\|gy_n - fz_n\| = d(fz_n, C). \qquad (*)$$

Let $A = \overline{f(C)}$ and $D = P(A)$. By ii) D is a compact subset of C. Since g is a proper map $g^{-1}(D)$ is a compact subset of C.

Now, for each n,

$$y_n \in F(z_n) = g^{-1}Pf(z_n) \subseteq g^{-1}Pf(C) \subseteq g^{-1}(D).$$

Consequently, there exists $y \in C$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \to y$. By $(*)$ we get that

$$\|g(y) - f(z)\| = d(fz, C).$$

This gives $y \in F(z) \cap B$, that is, $z \in F^{-1}(B)$ and F is an upper semicontinuous multifunction.

Now, the map $F = g^{-1} \circ P \circ f : C \to C$ is an admissible map. Since $f(C)$ is relatively compact therefore $F(C) = g^{-1}(P(f(C)))$ is also relatively compact because the image of a compact set under upper semicontinuous map with compact values is compact.

Then F satisfies conditions of Theorem 3 and has a fixed point, say $y_0 \in F(y_0)$. This implies that

$$\|gy_0 - fy_0\| = d(fy_0, C).$$

An almost affine map satisfies the condition that $g^{-1}(z)$ is an acyclic set for each $z \in C$. As a consequence we have the following [10].

COROLLARY 1. Let C be a nonempty convex subset of a normed linear space X and $P : X \to 2^C$ the metric projection satisfying i) and ii) of Theorem 4. Let $g : C \to C$ be a continuous, onto, almost affine and proper map. Then for each continuous map $f : C \to X$ with $f(C)$ relatively compact there exists a $y_0 \in C$ such that

$$\|gy_0 - fy_0\| = d(fy_0, C).$$

In case C is an approximatively compact set then conditions i) and ii) are satisfied and we have the following [3].

COROLLARY 2. Let C be an approximatively compact convex set of a normed linear space X and $g : C \to C$ continuous, onto, proper and $g^{-1}(z)$ is an acyclic subset of C for every $z \in C$. Then for each continuous function $f : C \to X$ with $f(C)$ relatively compact there exists an $x_0 \in C$ such that

$$\|gx_0 - fx_0\| = d(fx_0, C).$$

In case C is an approximatively compact and $g = I$, an identity function, then we get a well-known result of Reich [9].

Recently in [2] almost quasi convex g was considered.

The function $g : C \to X$ is said to be almost quasi convex if

$$\|g(\lambda x_1 + (1-\lambda)x_2) - y\| \leq \max(\|gx_1 - y\|, \|gx_2 - y\|)$$

for x_1, $x_2 \in C$, $y \in X$ and $0 < \lambda < 1$.

As in [3] it is easy to see that almost quasi convex implies that $g^{-1}(z)$ is an acyclic set. Therefore a recent result due to Park, Singh and Watson [6] given below, follows as a corollary.

COROLLARY 3. Let C be a nonempty convex subset of a normed linear space X and $P : X \to 2^C$ the metric projection satisfying

i) $Px \neq \emptyset$ for each $x \in X$, and

ii) P sends compact subset of X onto compact subsets of C.

Let $g : C \to C$ be a continuous, almost quasi convex, onto and proper map.

Then for each continuous map $f : C \to X$ with $\overline{f(C)}$ compact there is a $y_0 \in C$ such that

$$\|gy_0 - fy_0\| = d(fy_0, C).$$

Note. They [6] concluded that either

i) f and g have a coincidence point $w \in C$, i.e., $fw = gw$, or

ii) there is a $w \in C$ such that $gw \in \partial C$ and

$$0 < \|gw - fw\| \leq \|z - fw\| \quad \text{for all} \quad z \in I_C(gw).$$

Recall that if $f : C \to X$ is a map, then the inward set $I_C(x)$, of C at x is defined by

$$I_C(x) = \{y : y \in X, \ \text{there exist} \ u \in C \ \text{and} \ r > 0 \ \text{such that} \ y = x + r(u - x)\}.$$

The closure is denoted by $\overline{I_C(x)}$. The function f is said to be inward map if $f(x) \in I_C(x)$ for every $x \in C$ and weakly inward if $f(x) \in \overline{I_C(x)}$.

In case C is a compact convex subset of a normed linear space X we get the following.

COROLLARY 4. Let C be a compact convex subset of a normed linear space X and $g : C \to C$ continuous, onto, $g^{-1}(z)$ is an acylic set for each $z \in C$. Then for each continuous function $f : C \to X$ there is a $y_0 \in C$ such that

$$\|gy_0 - fy_0\| = d(fy_0, C).$$

If $g = I$, an identity function, in Corollary 4, then we get a well-known result of Ky Fan [4].

If in Corollary 4, g is almost-affine, continuous, onto then we get a theorem of Prolla [8].

If $fy_0 \in C$ for all y_0 then Corollary 4 yields a coincidence theorem, that is, there is a $y_0 \in C$ such that

$$gy_0 = fy_0.$$

In case g is an identity function in Corollary 4 and f has an additional condition then we get a fixed point theorem given below.

COROLLARY 5. If all the hypotheses of Corollary 4 are satisfied with $g = I$, and in addition: for any $x \in \partial C$ with $x \neq fx$ there exists a number λ (real or complex depending on whether the vector space is real or complex) such that $|\lambda| < 1$ and

$$y = \lambda x + (1 - \lambda) f x \in C\,,$$

then f has a fixed point.

PROOF. By Corollary 4 we have that

$$\|y_0 - fy_0\| = d(fy_0, C)\,.$$

Take $y_0 \in \partial C$ and assume that $y_0 \neq fy_0$. Then

$$\begin{aligned} 0 < \|y_0 - fy_0\| \leq \|y - fy_0\| &= \|\lambda y_0 + (1-\lambda) f y_0 - f y_0\| \\ &= \|\lambda (y_0 - fy_0)\| \\ &= |\lambda|\, \|y_0 - fy_0\| < \|y_0 - fy_0\| \end{aligned}$$

a contradiction. Hence $y_0 = fy_0$.

In case $g = I$, an identity function, then we could derive a very well-known result due to Browder [1] as a corollary.

COROLLARY 6. Let X be a Banach space, C a compact, convex subset of X and $f : C \to X$ a continuous function. Suppose that for each $x \in C$ with $x \neq fx$ there exists a $y \in I_C(x)$ such that

$$\|y - fx\| < \|x - fx\|\,.$$

Then f has a fixed point.

PROOF. By Corollary 4 (with $g = I$), we have

$$\|x - fx\| = d(fx, C)$$

i.e. $\|x - fx\| \leq \|fx - z\|$ for all $z \in C$.

This inequality remains valid (see [5]) for all $y \in I_C(x)$, i.e.

$$\|x - fx\| \leq \|fx - y\| \qquad \text{for all} \qquad y \in I_C(x)\,. \tag{$**$}$$

By hypothesis for each $x \in C$ with $x \neq fx$ we have $y \in I_C(x)$ such that

$$\|y - fx\| < \|x - fx\|$$

a contradiction to $(**)$. Hence f has a fixed point.

Further work in this direction due to Park [5] gives several applications.

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