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APPLICATION ON LOCAL DISCRETE EXPANSION

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ABSTRACT. The process of changing a topology by some types of its local discrete expansion preserves s-closeness, S-closeness, semi-compactness, semi-T_i, semi-R_i, $i \in \{0,1,2\}$, and extremely disconnectness Via some other forms of such above replacements one can have topologies which satisfy separation axioms the original topology does not have

KEY WORDS AND PHRASES: Near open sets, local discrete expansion, extremely disconnected, semi-compact, s-closed, S-closed, semi-T_i, semi-R_i, and cid spaces
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1. INTRODUCTION

Throughout the present paper (X,τ) is a topological space (or simply a space X) on which no separation axioms are assumed unless explicitly stated. For any $B \subset X$, $cl_\tau B$ (resp $\text{int}_\tau B$) denotes the closure (resp interior) of B A subset B is said to be regular open (resp regular closed) if $B = \text{int}_\tau(cl_\tau(B))$ (resp $B = cl_\tau(\text{int}_\tau(B))$) A subset B of a space X is said to be τ-semi open [12] (resp τ-regular semi-open [2]) if there exists a τ-open (resp. τ-regular open) set U satisfying $U \subset B \subset cl_\tau U$ B is τ-semi-closed [3] if the set $X - B$ is τ-semi-open. The family of all regular open (resp regular semi-open, semi-open) sets in X is denoted by $RO(X,\tau)$ (resp $RSO(X,\tau), SO(X,\tau)$) The union (resp intersection) of all τ-semi-open (resp τ-semi-closed) sets contained in B (resp containing B) is called the τ-semi-interior [3] (resp τ-semi-closure [3]) of B, and it is denoted as $s\text{-int}_\tau B$ (resp $s - cl_\tau B$) A space X is said to be extremely disconnected (denoted by E.D) if for every open set U of X, $cl_\tau U$ is open in τ The concept of local discrete expansion of a topology was first introduced by S P Young in 1977 [17], "Let (X,τ) be a topological space and A be any subset of X The topology $\tau[A] = \{U - H : U \in \tau, H \subset A\}$ is called the local discrete expansion of τ by A A space X is semi-T_2 [13] (resp semi-T_2' [1]) iff for $x, y \in X$, $x \neq y$ there exist U and $V \in SO(X,\tau), x \in U$ and $y \in V$ such that $U \cap V = \phi$ (resp $cl_\tau U \cap cl_\tau V = \phi$). Semi-$T_0$ and semi-T_1 were introduced to topological spaces [13] by replacing the word "open" by "semi-open" in the definitions of T_0 and T_1 respectively A space X is semi-R_0 [6] iff for each semi-open set U and $x \in U$, $s - cl_\tau\{x\} \subset U$ A space X is semi-R_1 [6] iff for $x, y \in X$ such that $s - cl_\tau\{x\} \neq s - cl_\tau\{y\}$ there exist disjoint semi-open sets U and V such that $s - cl_\tau\{x\} \subset U$, and $s - cl_\tau\{y\} \subset V$. A space X is called cid [15] if every countable infinite subspace of X is discrete. A space X is semi-compact [7] (resp s-closed [5], S-closed [16]) if for every cover $\{V_i : i \in I\}$ of X by semi-open sets of X, there exists a finite subset I_0 of I such that $X = \cup\{V_i : i \in I_0\}$ (resp $X = \cup\, scl(V_i) : i \in I_0\}, X = \cup\, cl(V_i) : i \in I_0\}$).

REMARK 1.1. For a subset A of a space (X,τ) we say that A satisfies condition (C_1) if $A \cup U = \phi$, for every $U \in \tau - \{X\}$.

Listed below are theorems that will be utilized in this paper

THEOREM 1.1 [14] If τ and τ' are two topologies on X such that $\tau \subset \tau'$, then $RO(X,\tau) = RO(X,\tau')$ iff $cl_\tau G = cl_{\tau'} G$ for every $G \in \tau'$ [equivalent iff $\text{int}_\tau F = \text{int}_{\tau'} F$, for every $F \in \tau'^c$]

THEOREM 1.2 [11] If X is a space, and $A \subset X$ satisfying (C_1) Then, $cl_{\tau[A]} G = cl_\tau G$, for every $G \in \tau[A]$

THEOREM 1.3 [4] If X is a space, and $A \in SO(X,\tau)$ such that $A \subset B \subset cl_\tau A$ Then, $B \in SO(X,\tau)$

THEOREM 1.4 [10] If X is a space, and $B \subset X$, then $s - cl_\tau B = B \cup \text{int}_\tau cl_\tau B$

THEOREM 1.5 [8] A space X is E D iff for every pair U and V of disjoint τ-open sets, we have $cl_\tau U \cap cl_\tau V = \phi$

THEOREM 1.6 [5] A space X is s-closed iff every cover of X by regular semi-open sets has a finite subcover

THEOREM 1.7 [15] (a) A space X is cid if every countable infinite subset is closed

(b) Any infinite cid space is T_1

THEOREM 1.8 [17] Let A be any subset of X Then $(A, \tau[A] \cap A)$ is discrete

THEOREM 1.9 [17] Let A be a closed subset of X Then $(A, \tau \cap A)$ is a discrete subspace of X iff $\tau = \tau[A]$

THEOREM 1.10 [9] Let X be a T_1-space Then X is cid iff countable subsets have no limits points

2. ON LOCAL DISCRETE EXPANSION

THEOREM 2.1. If (X,τ) is a space and $A \subset X$, then

(i) $SO(X,\tau[A]) \subset \{B - H : B \in SO(X,\tau), H \subset A\}$

(ii) If A satisfying (C_1), then the inclusion symbol in (i) is replaced by equality sign

PROOF. (i) Let $W \in SO(X,\tau[A])$, then there exists $V \in \tau[A]$ such that $V \subset W \subset cl_{\tau[A]}V$ Then $(U - H_1) \subset W \subset cl_{\tau[A]}(U - H_1)$, where $U \in \tau$, $H_1 \subset A$ Put $H_2 = U \cap H_1$, then $H_2 \subset A$, and $(U - H_1) \cup H_2 \subset W \cup H_2 \subset cl_{\tau[A]}(U - H_1) \cup H_2$ Then $U \subset W \cup H_2 \subset cl_{\tau[A]}U \subset cl_\tau U$, and $(W \cup H_2) \in SO(X,\tau)$ Put $B = W \cup H_2$, and $H = H_1 - W \subset A$ Then $B - H = W \cup (U \cap H_1) - (H_1 - W) = W$.

(ii) By Theorem 1.2, the proof is obvious

REMARK 2.1. From Theorem 2.1, it is easy to prove that, for any $A \subset X$

$SO(X,\tau) \subset SO(X,\tau[A])$

THEOREM 2.2. If (X,τ) is a space, and $A \subset X$ satisfying (C_1) Then

(i) $SO(X,\tau) = SO(X,\tau[A])$.

(ii) $RSO(X,\tau) = RSO(X,\tau[A])$.

PROOF. In general $SO(X,\tau) \subset SO(X,\tau[A])$. To prove the converse, let $W \in SO(X,\tau[A])$, then there exists $V \in \tau[A]$ satisfying $V \subset W \subset cl_{\tau[A]}V$. Then $(U - H) \subset W \subset cl_{\tau[A]}(U - H)$, $U \in \tau, H \subset A$. There are two cases.

(a) $U \neq X$, then $U - H = U$ Since $cl_{\tau[A]}U = cl_\tau U$, then $W \in SO(X,\tau)$.

(b) $U = X$, then $(X - H) \subset W \subset cl_{\tau[A]}(X - H) \subset cl_\tau(X - H)$. Since $A \cap U = \phi$, then $cl_\tau A \subset (X - U)$, and $cl_\tau A \cap U = \phi$, implies to $cl_\tau H \cap U = \phi$, for each $U \in \tau - \{X\}$ Hence $U \not\subset cl_\tau H$, and $\text{int}_\tau cl_\tau H = \phi$, and H is a τ-semi-closed set Thus $(X - H) \in SO(X,\tau)$ From Theorem 1 3, $W \in SO(X,\tau)$

(ii) By Theorems 1.1 and 1 2, the proof is obvious

COROLLARY 2.1. If X is a space, and $A \subset X$ satisfying (C_1) Then

(i) (X,τ) is semi-T_i iff $(X,\tau[A])$ is semi-T_i $(i \in \{0,1,2\})$.

(ii) If (X,τ) is semi-T_2', then $(X,\tau[A])$ is semi-T_2'.

(iii) If (X,τ) is semi-R_i, then $(X,\tau[A])$ is semi-R_i $(i \in \{0,1\})$

PROOF. By Theorem (2 2), the proof is obvious

THEOREM 2.3. If X is a space, and $A \subset X$ satisfying (C_1). Then $s - cl_{\tau[A]}G = s - cl_\tau G$, for every $G \in \tau[A]$

PROOF. Let $G \in \tau[A]$, then $s - cl_{\tau[A]}G = G \cup \text{int}_{\tau[A]}\, cl_{\tau[A]}G = G \cup \text{int}_\tau cl_{\tau[A]}G = G \cup \text{int}_\tau cl_\tau G = s - cl_\tau G$ [by Theorems 1 1, 1 2 and 1 4]

THEOREM 2.4. If X is a space, and $A \subset X$ satisfying (C_1). Then (X, τ) is E.D. iff $(X, \tau[A])$ is E D

PROOF. Let (X, τ) be E.D., $W \in \tau[A]$ Then $W = U - H, U \in \tau, H \subset A$.

But $cl_{\tau[A]}(U - H) = cl_{\tau[A]}U = cl_{\tau}U$, and $cl_{\tau}U \in \tau$. Thus $cl_{\tau[A]}W \in \tau[A]$, and $(X, \tau[A])$ is E.D Conversely, let $(X, \tau[A])$ be E.D., and $U, V \in \tau$ such that $cl_{\tau}U \cap cl_{\tau}V \neq \phi$. By Theorem 1.2, $cl_{\tau[A]}U \cap cl_{\tau[A]}V \neq \phi$, then $U \cap V \neq \phi$ [by Theorem 1.5]. Hence (X, τ) is E.D.

THEOREM 2.5. If X is a space, and $A \subset X$ satisfying (C_1). Then (X, τ) is semi-compact (resp s-closed) iff $(X, \tau[A])$ is semi-compact (resp. s-closed).

PROOF. By Theorem 2.2, the proof is obvious.

THEOREM 2.6. If X is a space, and $A \subset X$, and $(X, \tau[A])$ is S-closed (resp. s-closed), then (X, τ) is S-closed (resp. s-closed).

PROOF. Since $SO(X, \tau) \subset SO(X, \tau[A])$, the proof is obvious.

3. $L - T_i$ AND $Q - L - T_i$ SPACES

Let R be a topological property which is preserved under expansions

DEFINITION 3.1. A topological space (X, τ) is called $L - R$ if there exists a subset $S \subset X$ and $S \neq X$, such that $(X, \tau[S])$ has R.

PROPOSITION 3.1. If $\tau \subset \tau'$, then for any $S \subset X, \tau[S] \subset \tau'[S]$.

REMARK 3.1. If $\tau \subset \tau'$ and τ is $L - R$, then τ' is also $L - R$, i e. any expansion of $L - R$ topology on X is also $L - R$.

DEFINITION 3.2. Let $i = 1, 2, 2.5$ and $j = 0, 1, 2, 2.5$. We say that (X, τ) is $Q - L - T_i$, if it is $L - T_i$ and T_j where $j < i$.

Now we are going to show that some of the properties $L - T_i$ and $Q - L - T_i$ are satisfied for some spaces but not for some other spaces.

PROPOSITION 3.2. For a space X, the following diagram is easily obtained.

$$T_{2\frac{1}{2}} \Rightarrow Q - L - T_{2\frac{1}{2}} \Rightarrow T_2 \Rightarrow Q - L - T_2 \Rightarrow T_1 \Rightarrow Q - L - T_1 \Rightarrow T_0.$$

EXAMPLE 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a, b\}, \{c, d\}\}$ is not T_0 if $A = \{a, c\}$, then $\tau[A] = \{\phi, X, \{b\}, \{d\}, \{b, d\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, b, d\}\}$ is T_0. This example is $Q - L - T_0$.

The following is an example of a $Q - L - T_{2.5}$ but not $T_{2.5}$.

EXAMPLE 3.2. Let $X = N \times Z \cup \{(-1, 0), (-1, -1)\}$ where N is the natural numbers and Z the integers. The topology has as its base sets of the following forms:

$$\{(m, n)\}, \quad n \neq 0, \quad m \neq -1$$

$$U_n((a, 0)) = \{(a, 0)\} \cup \{(a, m) \,\big|\, |m| \geq n\}, \quad n \epsilon N$$

$$U_n((-1, 1)) = \{(-1, 1)\} \cup \{(a, m) \,\big|\, a \geq n, m > 0\}, \quad n \epsilon N$$

$$U_n((-1, -1)) = \{(-1, -1)\} \cup \{(a, m) \,\big|\, a \geq n, m < 0\}, \quad n \epsilon N.$$

This space is T_2 but not $T_{2.5}$ as $(-1, 1)$ and $(-1, -1)$ do not have disjoint closed neighborhoods. Choosing $A = N \times (Z - \{0\})$, the discrete expansion is the discrete topology and thus T_2.

EXAMPLE 3.3. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{b\}, \{d\}, \{b, d\}, \{a, b\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}\}$, then $\tau[A] =$ Discrete. This example is $Q - L - T_1$ but not T_1 and is an example of a space which is not $Q - L - T_2$.

EXAMPLE 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a, b\}\}$. If $A = \{a, b\}$, then $\tau[A] =$ Discrete This example is not $Q - L - T_1$.

The excluded point topology on an infinite set X is the family consisting of ϕ and all subsets of X not containing a point p of X.

EXAMPLE 3.5. The excluded point topology is $L - T_1$ and not $L - T_2$ (also is an example of $Q - L - T_1$ but not T_1).

PROOF. If X is an infinite set and p is the excluded point and $A \subset X$, then:

(i) If $p \notin A$, we have $\tau[A] = \tau \cup \{X - B : B \subset A\}$. Thus $\tau[A]$ is T_1 but not T_2.

(ii) If $p \in A$, then A is closed, and there are two cases

(a) If $B \subset A, p \in B$ in this case any open set in $\tau[A]$ is open in τ, i e $\tau = \tau[A]$

(b) If $B \subset A$, $p \notin B$ as (i) Thus $\tau[A] = \tau \cup \{X - B \cdot B \subset A\}$

EXAMPLE 3.6. Let $X = [0,1]$ and $\tau = \{\phi, X, A \subset X \cdot X - A \text{ is finite}\}$ If we take $S = (0,1]$, then $\tau[S]$ is the Discrete space This example is $Q - L - T_2$ but not T_2

THEOREM 3.1. (X,τ) is cid space iff $\tau = \tau[A]$ whenever A is a countable infinite subset of X

PROOF. We assume that (X,τ) is cid, then A is closed and discrete subspace By Theorem 1 9 we have that $\tau = \tau[A]$ Conversely we assume that $\tau = \tau[A]$ By Theorem 1 8, we have that $(A, \tau \cap A)$ is a discrete subspace of X and (X,τ) is cid space

THEOREM 3.2. Every space (X,τ) is $L - T_0$

PROOF. Assume that $x_0 \in X$ We aim to prove that $\tau[X - \{x_0\}]$ is T_0 For this purpose let $x, y \in X, x \neq y$, if $U \in \tau$ is an open set containing x, then $U - \{y\}$ is an open set in $\tau[X - \{x_0\}]$ and not containing y If $x_0 = x$, then $X - \{y\}$ is an open in $\tau[X - \{x_0\}]$ and not containing y This completes the proof

The following example illustrates a $Q - L - T_2$ space but not T_2

EXAMPLE 3.7. (Countable complement topology [16]) If X is an uncountable set, we define the topology of countable complements on X by declaring open all sets whose complements are countable, together with ϕ and X (X,τ) is T_1 but not T_2 Let $A \subset X$ such that $X - A$ is countable For $x_0 \in X - A$, $A \cup \{x_0\}$ is τ-open, and so $(A \cup \{x_0\}) - A = \{x_0\} \in \tau[A]$ For $x_0 \in A$, A is τ-open, which means that $A - (A - \{x_0\}) = \{x_0\}$ is $\tau[A]$-open Thus $\tau[A]$ is discrete and consequently T_2

UNSOLVED PROBLEM. If (X,τ) is a space which does not have a property P, what are the properties of the subset A that make $(X, \tau[A])$ have P (for P = fixed property)

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