Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 473-480

MONOTONE AND CONVEX H^*-ALGEBRA VALUED FUNCTIONS

PARFENY P. SAWOROTNOW

Department of Mathematics
The Catholic University of America
Washington, DC 20064
United States of America

(Received March 17, 1994 and in revised form May 19, 1995)

ABSTRACT. Classical theorems about monotone and convex functions are generalized to the case of H^*-algebra valued functions.

Also there are new examples of a vector measure.

KEY WORDS AND PHRASES. H^*-algebra, trace-class, Hilbert Schmidt operators, monotone functions, saltus function, convex function, Stieltjes integral, Borel measure.
1991 AMS SUBJECT CLASSIFICATION CODES. Primary: 26A48, 26A42, 26A51, 46K15 Secondary: 47B10, 46G10, 28A25, 26A46

1. INTRODUCTION.

An interesting fact about H^*-algebras [1] is that it can be used to generalize many classical theorems. In particular, positive H^*-algebra valued measures behave very much like (positive) scalar measures. There is also a more general theory of monotone and convex functions, and there is a natural generalization of the Stieltjes integral. The present paper is devoted to these theories. A biproduct is an example of a positive vector measure. It is constructed using a monotone H^*-algebra valued function.

2. PRELIMINARIES.

H^*-algebras were introduced by W. Ambrose in the forties [1]. They can be used to characterize Hilbert-Schmidt operators on a Hilbert space. A proper H^*-algebra is a semi-simple Banach $\star$-algebra A whose norm $\| \ \|$ is a Hilbert space norm (i.e., it is defined in terms of a scalar product $(\ ,\ )$, $\| x \|^2 = (x, x)$, $x \in A$) such that its involution $x \to x^*$ satisfies the condition $(xy, z) = (y, x^*z) = (x, zy^*)$ for all $x, y, z \in A$. Two important examples of an H^*-algebra are the class (σc) of Hilbert-Schmidt operators and the group-algebra $L^2(G)$, where G is a compact group. Further examples could be constructed by taking all possible direct sums of given H^*-algebras.

The trace-class τA of an H^*-algebra A is defined as the set of all products xy of members of A. It corresponds to the trace-class (τc) [8] of operators (the class of nuclear operators) in the theory of Hilbert-Schmidt operators. It is a Banach algebra with respect to some norm $\tau(\ )$ which is related to the norm $\| \ \|$ of A by the equality $\tau(a^*a) = \| a \|^2$, $a \in A$. There is a natural order $\leq$ defined on τA (and A) : $a \leq b$ if $0 \leq (bx, x) - (ax, x)$ for all $x \in A$ $(a, b \in \tau A)$ (it corresponds to the ordering by the cone of the positive operators for the case of a trace-class (τc) of Schatten [8]). The algebra τA has a trace tr (a positive linear functional) which has the property that $tr\, a = \tau(a)$ if $a \geq 0$ and $tr\,(xy) = tr\,(yx) = (y^*, x)$ for all $x, y \in A$.

The norm $\tau(\ )$ of τA is additive on the set of positive members of τA : if $a, b \geq 0$ then $\tau(a+b) = tr\,(a+b) = tr\, a + tr\, b = \tau(a) + \tau(b)$ (it follows that $\tau(a) \leq \tau(b)$ if $0 \leq a \leq b$).

An important property of τA is the fact that it is monotone complete in the sense of Wright [10]: if E is a directed upward ($\forall x,\ y \in E\ \exists\ z \in E$ such that $x,\ y \leq z$) bounded above ($\exists\ m \in \tau A$ such that $x \leq m$ for all $x \in E$) subset of τA then $\exists\ m_0 \in \tau A$ such that $m_0 = \operatorname{lub} E$. One can also show [7, Lemma 2] that in this case

$$\underset{x \in E}{lub}\quad \tau(m_0 - x) = 0.$$

3. MONOTONE FUNCTIONS.

There is no simple characterization of the class of functions which are differences of increasing trace-class valued functions (unlike in the case of real valued functions). For this reason we shall restrict our theory to the case of monotone functions.

Let A be a proper H^*-algebra and let τA be its trace-class. An increasing τA-valued function $x(t)$ on the real line R is defined in the obvious way by the condition "$t < s$ implies $x(t) \leq x(s)$ $(0 \leq (x(s)a,\ a) - (x(t)a,\ a)$ for all $a \in A)$." Decreasing functions are defined similarly.

THEOREM 1. For each increasing τA-valued function $x(t)$ there are increasing functions $x(t)^+$ and $x(t)^-$ such that $x(t)^- = lub\{x(s) : s < t\}$ and $x(t)^+ = glb\{x(s) : s > t\}$. It is also true that

$$\lim_{s \to t^-} \tau(x(s) - x(t)^-) = 0$$

and

$$\lim_{s \to t^+} \tau(x(t)^+ - x(s)) = 0.$$

Similar statements hold for decreasing τA-valued functions.

PROOF. This is a consequence of Lemma 1 and Lemma 2 of [7]: for a fixed $t \in R$ the set $\{x(s) : s < t\}$ is directed upward, hence it has a least upper bound $x(t)^-$ which is also the left limit of $x(s)$ with respect to the norm of τA.

THEOREM 2. Let $x(t)$ be an increasing τA-valued function defined on R. For each $t \in R$ let $j(t) = x(t)^+ - x(t)^-$. Then the set $D = \{t \in R : j(t) \neq 0\}$ is countable (note that D is the set of discontinuities of $x(t)$).

PROOF. We use the fact that the norm τ of τA is additive on positive members of τA. Let α and β be any two real numbers with $\alpha < \beta$ and let $t_1 < t_2 < \ldots < t_n$ be some points in the open interval $(\alpha,\ \beta)$.

Then

$$\sum_{k=1}^{n} \tau j(t_k) = \sum_{k=1}^{n} \tau(x(t_k)^+ - x(t_k)^-) \leq \sum_{k=1}^{n} \tau(x(t_k)^+ - x(t_{k-1})^+) =$$
$$= \tau\left(x(t_n)^+ - x(t_0)^+\right) \leq \tau\left(x(\beta) - x(\alpha)\right)$$

(here t_0 is any point between α and t_1)
from which it follows that

$$\sum_{k=1}^{n} \tau(j(t_k)) \leq \tau(x(\beta) - x(\alpha)).$$

From this we conclude that for each positive integer p there is only a finite number of points t in $(\alpha,\ \beta)$ such that $\frac{1}{p} \leq \tau(j(t))$. Consequently the set D must be countable.

COROLLARY. Every monotone τA-valued function is continuous almost everywhere.

In analogy with [2, p. 14] we define a τA-valued saltus function in more or less the same fashion. Let $\{a_k\}$ and $\{b_k\}$ be some fixed sequences of positive members of τA such that

$$\sum_k \tau(a_k) < \infty \text{ and } \sum_k \tau(b_k) < \infty,$$

and let $T = \{t_k\}$ be a fixed countable set of distinct real numbers. The corresponding saltus function $\sigma(t)$ is defined by the formula:

$$\sigma(t) = \sum_{t_k \leq t} a_k + \sum_{t_k < t} b_k.$$

It is easy to see that the function $\sigma(t)$ is monotone, continuous for each $t \neq t_k$ and $\sigma(t_k)^+ - \sigma(t_k) = b_k$, $\sigma(t_k) - \sigma(t_k)^- = a_k$ for each t_k in the set T. The next theorem is a generalization of a well known theorem about increasing real valued functions.

THEOREM 3. Each increasing τA-valued function $x(t)$ on R can be represented as a sum $x(t) = c(t) + \sigma(t)$ of a continuous increasing function $c(t)$ and a saltus function $\sigma(t)$. If $x(t)$ is right (left) continuous then $\sigma(t)$ is also right (left) continuous.

PROOF. The proof is essentially the same as in the case of real valued monotone functions (e.g. [2, pp. 14–15]). We included it for the benefit of those who are not familiar with the classical theory. For each $t \in R$ define $a(t) = x(t) - x(t)^-$, $b(t) = x(t)^+ - x(t)$ and let $T = \{t \in R :$ both $a(t)$ and $b(t)$ are non-zeros $\}$. Then T is countable, and so it can be written as a sequence, $T = \{t_k\}$ (where k runs through natural numbers).
Define:

$$\sigma(t) = \sum_{t_k \leq t} a(t_k) + \sum_{t_k < t} b(t_k)$$

and

$$c(t) = x(t) - \sigma(t).$$

Let us show that $c(t)$ is right continuous. For any $s > t$ we have (each t_k belongs to T):

$$c(s) - c(t) = x(s) - x(t) - (\sigma(s) - \sigma(t)) = x(s) - x(t) - \sum_{t < t_k \leq s} a(t_k) - \sum_{t \leq t_k < s} b(t_k) =$$

$$= x(s) - x(t) - b(t) - \sum_{t < t_k \leq s} a(t_k) - \sum_{t < t_k < s} b(t_k) =$$

$$= x(s) - x(t)^+ - \sum_{t < t_k \leq s} a(t_k) - \sum_{t < t_k < s} b(t_k).$$

Thus:

$$\tau(c(s) - c(t)) \leq \tau(x(s) - x(t)^+) + \sum_{t < t_k \leq s} \tau(a(t_k)) + \sum_{t < t_k < s} \tau(b(t_k)).$$

Now we use the fact that

$$\sum_k \tau(a(t_k)) < \infty, \qquad \sum_k \tau(b(t_k)) < \infty.$$

For any $\varepsilon > 0$ we can select k_0 so that

$$\sum_{k \geq k_0} \tau(a(t_k)) < \frac{\varepsilon}{4}$$

and

$$\sum_{k \geq k_0} \tau(b(t_k)) < \frac{\varepsilon}{4}.$$

If we now select $\delta > 0$ so that $\tau(x(s) - x(t)^+) < \frac{\varepsilon}{4}$ if $s - t < \delta$ and so that the interval $[t, t + \delta]$ would exclude the points $t_1, t_2, \ldots, t_{k_0}$, then $0 < s - t < \delta$ would imply $\tau(c(s) - c(t)) < \varepsilon$. Left continuity of $c(t)$ is established in a similar fashion.

4. STIELTJES INTEGRAL.

Let μ be a positive τA-valued Borel measure on the real line (μ is a countably additive positive τA-valued set function defined on the σ-algebra [3] $\mathcal{B}$ generated by open sets of real numbers). For each real t define $x(t) = \mu[-\infty, t) = \mu\{s \text{ real} \mid s < t\}$. Then $x(t)$ is an increasing positive left continuous τA-valued function defined on R.

The converse is also true. Let $x(t)$ be as in the preceding paragraph. Let $f(t)$ be a bounded real valued function defined on some interval $[\alpha, \beta]$. We can now apply the classical procedure to define the Stieltjes integral

$$\int_\alpha^\beta f(t)\, dx(t).$$

Let $\wedge = \{\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta\}$ be a partition of $[\alpha, \beta]$, and let $m_i = glb\{f(t) : t_{i-1} \leq t \leq t_i\}$, $M_i = lub\{f(t) : t_{i-1} \leq t \leq t_i\}$ for $i = 1, 2, \ldots, n$. We define

$$s(\wedge) = \sum_{i=1}^{n} m_i \Delta x(t_i), \quad S(\wedge) = \sum_{i=1}^{n} M_i \Delta x(t_i)$$

where $\Delta x(t_i) = x(t_i) - x(t_{i-1})$. Then the set Γ_1 of all lower sums $s(\wedge)$ is an upward directed set, bounded above by a member of τA, and the set Γ_2 of all upper sums $S(\wedge)$ is a bounded below, by a member of τA, downward directed set (it is easy to verify that $s(\wedge_1) \leq S(\wedge_2)$ for any 2 partitions $\wedge_1$, $\wedge_2$ of $[\alpha, \beta]$ (e.g. one can modify the technique of [4, pp. 105–107]).

Thus there are members $\int_{-} f(t)\,dx(t)$ and $\int^{-} f(t)\,dx(t)$ of τA such that $\int_{-} f(t)\,dx(t) = lub\,\Gamma_1$ and $\int^{-} f(t)\,dx(t) = glb\,\Gamma_2$. It is always true that $\int_{-} f(t)\,dx(t) \leq \int^{-} f(t)\,dx(t)$.

We can define $f(t)$ to be Stieltjes integrable if $\int_{-} f(t)\,dx(t) = \int^{-} f(t)\,dx(t)$, and in which case we denote the common value by

$$\int_{\alpha}^{\beta} f(t)\,dx(t).$$

It is not difficult to see that a real valued continuous function is Stieltjes integrable.

Let us consider the positive linear functional

$$I(f) = \int_{\alpha}^{\beta} f(t)\,dx(t) = \int_{-\infty}^{\infty} f(t)\,dx(t)$$

defined on the class L of all continuous real valued functions, each vanishing outside of some (finite) interval included in $[\alpha, \beta]$ (in which case the value of the integral is independent of a particular choice of α and β). Applying to I the Daniell theory, developed in [7] we obtain a positive Borel measure μ on R such that $\mu[s, t) = \mu\{r : s \leq r < t\} = x(t) - x(s)$ for all s, t in R. From this it is not difficult to see that each interval of the form $[-\infty, t)$ is summable and $\mu[-\infty, t) = x(t) + a$ for some positive $a \in \tau A$.

We summarize the above theory in the next theorem.

THEOREM 4. For each increasing positive left continuous τA-valued function $x(t)$ on R so that $\lim_{x \to -\infty} x(t) = 0$ there exists a positive regular τA-valued Borel measure μ on R such that $\mu[-\infty, t) = x(t)$ for each $t \in R$. The integral $\int f(t)\,d\mu(t)$ (corresponding to the measure μ) is identical to the Stieltjes integral

$$\int_{\alpha}^{\beta} f(t)\,dx(t)$$

on the set of continuous real valued functions $f(t)$ vanishing outside of some interval $[\alpha, \beta]$ (for each choice of the interval $[\alpha, \beta]$).
(The integral $\int f(t)\,d\mu(t)$ above is considered in the same sense as the integral $\int f\,d\mu$ in Theorem 4 in [7]).

5. EXAMPLES.

A τA-valued saltus function considered above constitutes an example of a discrete monotone function. Now we shall discuss continuous (monotone) functions.

Let $f(t)$ be a τA-valued function such that

$$f'(t) = \lim_{h \to 0} h^{-1}(f(t+h) - f(t))$$

exists for each t and is positive. To be specific let $a_0, a_1, \ldots, a_n$ be positive members of τA and let

$$f(t) = \sum_{k=0}^{n} a_k t^{2k+1}.$$

Then it is easy to see that

$$f'(t) = \sum_{k=0}^{n} a_k (2k+1) t^{2k}$$

for each $t \in R$, and the last expression is everywhere positive in the sense that $(f'(t)a,\ a) \geq 0$ for each $a \in A$ (note that

$$(f'(t)a,\ a) = \sum_{k=1}^{n} t^{2k}(2k+1)(a_k a,\ a)).$$

But it is also easy to see that $\frac{d}{dt}(f(t)a,\ a) = (f'(t)a,\ a)$. From this we may conclude that $f(t)$ is an increasing function.

A decreasing continuous function is also easy to construct.

6. CONVEX FUNCTIONS.

Also there is a natural generalization of the theory of convex functions, as it was developed, for example, on pages 113–115 of the third edition of Royden's book [3]. Let A and τA be as above. One can define a τA-valued convex function $X(t)$ the same way [3, page 113], using the inequality "$X(\lambda t + (1-\lambda)s) \leq \lambda X(t) + (1-\lambda)X(s),\ 0 \leq \lambda \leq 1$". It is obvious, that for the case when each $X(t)$ is self-adjoint, this definition is equivalent to stating that "a τA-valued function $X(t)$ is convex if for each $a \in A$ the scalar function $\varphi_a(t) = (X(t)a,\ a)$ is real valued and convex (in the sense of, say, Royden [3, p. 113])."

Below we shall assume that $X(t)^* = X(t)$ for each t.

The function $f(t)$ in the above Example (in Section 5) is convex on $[0,\ \infty)$, since $(f''(t)a,\ a) \geq 0$ for all $a \in A$ and $t > 0$ (one can apply 19. Corollary on page 115 of [3] here).

It turns out that most of the properties of convex functions stated on pages 113–115 of [3] are valid also for τA-valued convex function (one should remark here that the notion of derivates, defined on p. 99 of [3], are not very useful in our case – so we shall not attempt to generalize 18. Proposition on page 114 of [3]).

Some of these properties could be derived from the classical statements (e.g., pp. 113–115 of [3]), the others need to be verified directly.

Let us use the notation $Q(t,\ s) = \frac{X(s)-X(t)}{s-t}$, where s and t are some real numbers and $X(\ )$ is a fixed τA-valued function defined on some open interval $(\alpha,\ \beta)$ (where α and β are either finite real numbers or $\pm\infty$).

LEMMA. If $X(\ )$ is convex and $t \leq t' < s,\ t < s \leq s'$, then $Q(t,\ s) \leq Q(t',\ s')$.

PROOF. This is a consequence of 16. Lemma on page 113 of [3]. For each $a \in A$ we apply this lemma to the scalar function $\varphi_a(t) = (X(t)a,\ a)$ in order to arrive at the inequality "$(Q(t,\ s)a,\ a) \leq Q(t, s')a,\ a)$."

THEOREM 5. If $X(t)$ is convex, then its right and left derivatives $X'_+(t)$ and $X'_-(t)$ exist for each t in $(\alpha,\ \beta)$ and are increasing functions. It follows that $X(t)$ is continuous. Moreover, it is also true that $X'_-(t) \leq X'_+(t)$ for each $t \in (\alpha,\ \beta)$. If $X(t)$ is increasing, then it is absolutely continuous at each closed subinterval $[\alpha',\ \beta']$ of $(\alpha,\ \beta)$.

PROOF. Existence of X'_+ and X'_- is a consequence of the above Lemma and monotone completeness (Corollary 2 on page 878 in [7]) (see also the last paragraph in Section 2 above). If $t_0 \in (\alpha,\ \beta)$ is fixed, then the set $\{Q(t_0,\ s) \mid s \in (\alpha,\ \beta)\}$ is a directed downward set, bounded below by some member $Q(t',\ s')$ of τA. One can now use Corollary 1 on page 878 of [7]. Continuity of $X(t)$ could be derived as in the classical case (right differentiability implies right continuity, a function is continuous if it is both right and left continuous).

Now assume that $X(t)$ increases. In this case each $Q(t,\ s)$ is positive. If $[\alpha',\ \beta'] \subset (\alpha,\ \beta)$, one can select $\alpha_1,\ \beta_1$ so that $\beta' < \alpha_1 < \beta_1 < \beta$. Then we have (because of the above Lemma): $0 \leq D(t,\ s) \leq D(\alpha_1,\ \beta_1)$ for all $t,\ s \in [\alpha',\ \beta']$ with $t < s$. Since the norm τ is additive on positive members of τA, we have $\tau D(t,\ s) \leq M$, where $M = \tau D(\alpha_1,\ \beta_1)$. It follows that $\tau(X(s) - X(t)) \leq M(s-t) = M \mid s-t \mid$ for all $s,\ t \in [\alpha',\ \beta']$ with $t < s$. Absolute continuity of $X(\ )$ is now easy to establish: if $\{[t_i,\ s_i]\}\ i = 1,\ \ldots,\ n$ are nonoverlapping intervals in $[\alpha',\ \beta']$, then

$$\sum_{i=1}^{n} \tau(X(s_i) - X(t_i) \leq \tau(X(s_n) - X(t_1)) \leq M \mid s_n - t_1 \mid =$$

$$= M\sum_{i=1}^{n}(s_i - t_i) = M\sum_{i-1}^{n} \mid s_i - t_i \mid .$$

THEOREM 6. If $X(t)$ is twice differentiable and $x''(t) \geq 0$, then $X(\ )$ is convex.

One can use this theorem to construct convex functions. If $a_1, \ldots, a_n \in \tau A$ are **positive**, then $X(t) = \sum_{k=1}^{n} a_k t^k$ is convex for $t \geq 0$.

Jensen Inequality is also valid for our case, as it is stated in the next theorem.

THEOREM 7. Let $X(u)$ be a τA-valued convex function, defined for each real u and let $f(t)$ be a real valued measurable function, finite everywhere on $(-\infty, \infty)$. Assume that both $f(t)$ and $X(f(t))$ are summable on the interval [0, 1]. Then

$$X\left(\int_0^1 f(t)dt\right) \leq \int_0^1 X(f(t))dt.$$

PROOF. Let u_0, u and u_1 be such that $u_0 < u < u_1$. Then it follows from above Lemma that

$$Q(u_0, u) \leq Q(u_0, u_1).$$

This inequality can be written as

$$X(u_0) + (u_1 - u_0)\,Q(u_0, u) \leq X(u_1).$$

Taking limit as u approaches u_0 we arrive at the inequality

$$X(u_0) + (u_1 - u_0)\,X'_+(u_0) \leq X(u_1)$$

Also, considering $u_1 < u < u_0$ we can arrive, in a similar fashion, at the inequality:

$$X(u_0) + (u_1 - u_0)\,X'_-(u_0) \leq X(u_1).$$

It follows that

$$X(\alpha) + (u - \alpha)\,X'_-(\alpha) \leq X(u)$$

for all u and α (since $X'_-(\alpha) < X'_+(\alpha)$.

If we let $\alpha = \int_0^1 f(t)$, $m = X'_-(u_0)$ and $u = f(t)$, we arrive at the inequality:

$$m(f(t) - \alpha) + X(\alpha) \leq X(f(t)).$$

Taking definite (Bochner) interal on both sides we obtain:

$$m\left(\int_0^1 f(t)dt - \int_0^1 \alpha\, dt\right) + \int_0^1 X(\alpha)dt \leq \int_0^1 X(f(t))dt$$

or

$$X\left(\int_0^1 f(t)dt\right) \leq \int_0^1 X(f(t))dt.$$

It is appropriate to remark that we need to assume that $X(f(t))$ is summable over [0, 1], since in the theory of Bochner integrals the idea of an integral assuming an infinite value does not make sense.

This was not the case for scalar functions, considered by Royden in [3]. The following example may illustrate the point.

Let $f(t) = t^{-1/2}$ and $X(u) = u^2$. Then $X(\ )$ is convex, $f(t)$ is summable over [0, 1], but $g(t) = X(f(H))$ is not $\left(\int_0^1 X(f(t))dt = \int_0^1 \frac{dt}{t} = \infty\right)$. However, the Jensen inequality is still valid. Royden did not have to assume that $\int_0^1 \varphi(f(t)dt)$ is finite.

ACKNOWLEDGEMENT. The author expresses his thanks to the referee for his helpful comments.

REFERENCES

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364–386.
2. Frigyes Riesz and Bela Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., 1955.
3. H.L. Royden, *Real Analysis*, The Macmillan Co., New York, 1988.
4. Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1964.
5. P.P. Saworotnow and J.C. Friedell, *Trace-Class of an arbitrary H^*-algebra*, Proc. Amer. Math. Soc. **26** (1970), 95–100.
6. P.P. Saworotnow, *A generalized Hilbert space*, Duke Math J. **35** (1968), 191–197.
7. P.P. Saworotnow, *Daniell construction for an H^*-algebra Valued Measure*, Indiana University Math. J. **25** (1976), 877–882.
8. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik und ihrer grenzgebiete, Heft 27, Springer-Verlag, Berlin, 1960.
9. David V. Widder, *Advanced Calculus*, Prentice-Hall, 1947.
10. J.D. Maitland Wright, *Measures with values in a partially ordered vector space*, Proc. London Math. Society (3) **25** (1972), 675–688.