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ON THE SHIFT OPERATORS

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ABSTRACT. The purpose of this paper is to show that the weighted s-shift operators and so the weighted shift and the right shift operators have the SVEP, but the left shift operator has not Also, if $T, S \in B(X)$ are quasi-similar operators then, it is shown that T has the SVEP iff S has the SVEP Finally, the paper shows that the right and left shift operators are not decomposable

KEY WORDS AND PHRASES. The shift operators, decomposable operators, single-valued extension property (SVEP)
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INTRODUCTION.

Throughout this paper, the following notations are used

$\mathbb{C}$ the complex plane, X-A complex Banach space, $B(X)$ - the class of all bounded linear operators on X,

$\sigma(T)$ the spectrum of $T \in B(X)$,

$\sigma_p(T)$ the point spectrum of $T \in B(X)$,

$\overline{A}$ the closure of A (in a given topological space),

A^0 the interior of A (in a given topological space),

T^* the adjoint of T,

T/Y the restriction of $T \in B(X)$ to the invariant subspace $Y \subset X$

1. THE SINGLE-VALUED EXTENSION PROPERTY (SVEP)

DEFINITION 1.1. [1]

$T \in B(X)$ is said to have the single-valued extension property (SVEP) if for every function $f : D(\subset \mathbb{C}) \to X$ analytic on the open set D, the condition

$$(\lambda - T) f(\lambda) = 0 \quad \text{on} \quad D \quad \text{implies} \quad f \equiv 0 .$$

If T has SVEP then for any $x \in X$, $\rho_T(x)$ will denote the maximal domain of existence of the analytic X-valued function $\tilde{x}$ such that $(\lambda - T)\tilde{x}(\lambda) = x$, and the complement of $\rho_T(x)$ will be denoted by $\sigma_T(x)$ and it is called the local spectrum of T at x

If T has the SVEP then for any closed set $F \subset \mathbb{C}$, we put

$$X_T(F) = \{x : x \in X \quad \text{and} \quad \sigma_T(x) \subset F\} .$$

T Yoshino [5] proved that $T \in B(x)$ has the SVEP if $\sigma_p^0(T) = \phi$ Now, consider the Hilbert space l_2 of all square-summable sequences, i e ,

$$x = (x_i)_1^\infty \quad \text{and} \quad \sum_{i=1}^{\infty} |x_i|^2 < \infty .$$

DEFINITION 1.2. [4]

Let s be an integer greater than 0 and let $(\sigma_n)_1^\infty$ be an arbitrary sequence of non-zero complex numbers An operator $T \in B(l_2)$ is said to be a weighted s-shift with weights $(\sigma_n)_1^\infty$ if there exists an orthonormal basis $(e_n)_1^\infty$ of l_2 such that

$$Te_n = \sigma_n e_{n+s} , \quad n = 1, 2, 3, \dots .$$

Note that if $x \in l_2$ then $x = (x_1, x_2, x_3, \dots)$ and

$$Tx = (0, 0, \dots, 0, \sigma_1 x_1, \sigma_2 x_2, \sigma_3 x_3, \dots) .$$

THEOREM 1.1.

If $T \in B(l_2)$ is a weighted s-shift operator with weights $(\sigma_n)_1^\infty$, then $\sigma_p(T) = \phi$ and hence T has the SVEP

PROOF.

Let $\lambda \in \sigma_p^0(T)$, then there exists $0 \neq x \in l_2$ such that $x = (x_1, x_2, \dots)$ and

$$Tx = \mu x \quad \text{for all} \quad \mu \in D_r(\lambda)$$

where

$$D_r(\lambda) = \{\mu : |\mu - \lambda| < r, \ r > o\} .$$

Hence,

$$(0, 0, \dots, 0, \sigma_1 x_1, \sigma_2 x_2, \dots) = (\mu x_1, \mu x_2, \mu x_3, \dots)$$

and so

$$\mu x_m = 0 , \quad m = 1, 2, \dots, s \quad \text{and} \quad \mu x_{n+s} = \sigma_n x_n , \quad n = 1, 2, \dots .$$

If $\mu = 0$ then $x_n = 0$ for all $n(\sigma_n \neq 0)$. If $\mu \neq 0$ then $x_m = 0$, $m = 1, 2, \dots, s$ and $\mu x_{s+1} = \sigma_1 x_1 = 0$ which implies $x_{s+1} = 0$

In the same manner, we show $x_{s+n} = 0$, $n = 2, 3, \dots$. Therefore $x = 0$ and this contradicts that $x \neq 0$ Hence, $\sigma_p(T) = \phi$ and T has the SVEP

COROLLARY 1.1.

V I. Istratescue [3] defined the weighted shift operators as: $S \in B(l_2)$ is called weighted shift with the weight sequence $(W_n)_1^\infty$ if

$$S(x_1, x_2, x_3, \dots) = (0, W_1 x_1, W_2 x_2, \dots) .$$

It is clear that weighted 1-shifts coincide with weighted shifts with non-zero weight sequence. Hence, by Theorem 1 1, every weighted shift operator with non-zero weight sequence $(W_n)_1^\infty$ has the SVEP.

PROPOSITION 1.1. [1]

Let H be a Hilbert space, if $T \in B(H)$ is an isometric non-unitary operator then T^* has not the SVEP

COROLLARY 1.2.

The right shift operator $R \in B(l_2)$ is defined by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) .$$

It is clear that the right shift operators coincide with weighted l-shifts with weights $(1)_1^\infty$ Hence, by Theorem 1 1 , every right shift operator has the SVEP

COROLLARY 1.3.

The left shift operator $L \in B(l_2)$ is defined by

$$L(x_1, x_2, x_3, ...) = (x_2, x_3, x_4, ...) \ .$$

Note that $R^* = L$. Since R is an isometric non-unitary operator (see [2]) and $L = R^*$, then, by Proposition 1 1, every left shift operator has not the SVEP

THEOREM 1.2.

Let T be a weighted s-shift operator on l_2 If G is an open set such that $G \subset \sigma(T)$ and $0 \notin \overline{G}$, then $X_T(\overline{G}) = \{0\}$

PROOF.

Let $x \in X_T(\overline{G})$ then $\sigma_T(x) \subset \overline{G}$, since $o \notin \overline{G}$, we have $0 \in \rho_T(x)$ and hence, there is an analytic function $f : V_0 \to l_2$ such that

$$(\mu - T)f(\mu) = x \quad \text{on} \quad V_0, ... \tag{1.1}$$

where V_0 is a neighborhood of 0 Since, f is analytic on V_0 and $f(\mu) \in l_2$, then

$$f(\mu) = (f_1(\mu), f_2(\mu), ...) \ ,$$

where $f_n : V_0 \to \mathbb{C}$ is analytic on V_0 for all n By (1 1), we have

$$\mu f_m(\mu) = x_m \ , \quad m = 1, 2, ..., s$$

and

$$\mu f_{s+n}(\mu) - \sigma_n f_n(\mu) = x_{s+n} \ , \quad n = 1, 2, ...$$

since $0 \in V_0$ then $x_m = 0$, $m = 1, 2, ..., s$ let $\mu \neq 0$ then we have

$$f_1(\mu) = f_2(\mu) = ... = f_s(\mu) = 0 \ .$$

Hence,

$$\mu f_{s+1}(\mu) - \sigma_1 f_1(\mu) = x_{s+1}$$

which implies that $f_{s+1}(\mu) = x_{s+1}/\mu$ since f_{s+1} is analytic at 0 then $x_{s+1} = 0$ and so $x_{s+2} = x_{s+3} = ... = 0$. Hence $x = 0$ which proves that

$$X_T(\overline{G}) = \{0\} \ .$$

DEFINITION 1.3. [3]

$T, S \in B(X)$ are called quasi-similar if there exist injective operators $P, Q \in B(X)$ with dense ranges and such that:

(i) $TP = PS$;

(ii) $QT = SQ$.

THEOREM 1.3.

If $T, S \in B(X)$ are quasi-similar then T has the SVEP iff S has the SVEP

PROOF.

Since $T, S \in B(X)$ are quasi-similar then there exist $P, Q \in B(X)$ such that

$$TP = PS \quad \text{and} \quad QT = SQ \ .$$

Now, let T have the SVEP and $(\lambda - S)f(\lambda) = 0$ where $f : D \to X$ is an analytic function on D Then $P(\lambda - S)f(\lambda) = 0$, which implies that $(\lambda - T)Pf(\lambda) = 0$, since $Pf : D \to X$ is analytic and T has the SVEP, then $Pf(\lambda) = 0$ By the injectivity of P, we have $f(\lambda) = 0$ and S has the SVEP

Conversely, let S have the SVEP and $(\mu - T)g(\mu) = 0$, where $g : G \to X$ is analytic on G Then, by the same manner above and $QT = SQ$, T has the SVEP

2. DECOMPOSABLE OPERATORS

Given $T \in B(X)$, an invariant subspace Y is called the spectral maximal space of T if for any invariant subspace Z, the inclusion

$$\sigma(T/Z) \subset \sigma(T/Y)$$

implies $Z \subset Y$ Denote by $SM(T)$ the family of spectral maximal spaces of T.

DEFINITION 2.1. [1]

The operator $T \in B(X)$ is called decomposable if, for any open cover $\{G_i\}_1^n$ of $\sigma(T)$, there is a system $\{Y_i\}_1^n \subset SM(T)$ such that

(i) $\sigma(T/Y_i) \subset G_i, \quad 1 \leq i \leq n,$

(ii) $X = \sum_{i=1}^{n} Y_i$

PROPOSITION 2.1. [1]

If $T \in B(X)$ is decomposable then $\sigma_p^0(T) = \phi$; i.e , T has the SVEP

PROPOSITION 2.2. [1]

If $T \in B(X)$ is decomposable and $F \subset \sigma(T)$ is a closed set such that $X_T(F) = \{0\}$, then F has no interior point in $\sigma(T)$

COROLLARY 2.1.

The right and left shift operators are not decomposable.

PROOF.

Let T be a right shift operator and G an open set such that $G \subset \sigma(T)$ and $0 \notin \overline{G}$.

Since T is a weighted 1-shift with weights $\{1\}_1^\infty$ then, by Theorem 1.2, we have

$$X_T(\overline{G}) = \{0\} . \tag{2.1}$$

Now, since $\sigma(T) = \{\lambda : |\lambda| \leq 1\}$ (see [2]), and $F = \overline{G} \subset \sigma(T)$ is a closed set, we get.

$$F^o \cap \sigma(T) \neq \phi . \tag{2.2}$$

Therefore, by (2.1), (2.2) and Proposition 2.2, we have T is not decomposable. Finally, let S be a left shift operator. Then by Corollary 1 3, S has not the SVEP. Hence, by Proposition 2.1, S is not decomposable

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