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ON A CLASS OF EXACT LOCALLY CONFORMAL COSYMLECTIC MANIFOLDS

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ABSTRACT. An almost cosymplectic manifold M is a $(2m+1)$-dimensional oriented Riemannian manifold endowed with a 2-form Ω of rank $2m$, a 1-form η such that $\Omega^m \wedge \eta \neq 0$ and a vector field ξ satisfying $i_\xi\Omega = 0$ and $\eta(\xi) = 1$. Particular cases were considered in [3] and [6].

Let (M, g) be an odd dimensional oriented Riemannian manifold carrying a globally defined vector field T such that the Riemannian connection is parallel with respect to T. It is shown that in this case M is a hyperbolic space form endowed with an exact locally conformal cosymplectic structure. Moreover T defines an infinitesimal homothety of the connection forms and a relative infinitesimal conformal transformation of the curvature forms.

The existence of a structure conformal vector field C on M is proved and their properties are investigated. In the last section, we study the geometry of the tangent bundle of an exact locally conformal cosymplectic manifold.

KEY WORDS AND PHRASES: Locally conformal cosymplectic manifold, T-parallel connection, infinitesimal homothety, infinitesimal conformal transformation, Hamiltonian vector field, tangent bundle, Liouville vector field, complete lift, mechanical system.
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1. INTRODUCTION

In the last decade a series of papers have been devoted to almost cosymplectic manifolds $M(\Omega, \eta, \xi, g)$. As is well known, an almost cosymplectic manifold M is an odd dimensional (say $2m+1$) oriented manifold, where the triple (Ω, η, ξ) of tensor fields is

i) a 2-form Ω of rank $2m$

ii) a 1-form η such that $\Omega^m \wedge \eta \neq 0$

iii) a vector field (called the Reeb vector field) such that $i_\xi\Omega = 0$ and $\eta(\xi) = 1$.

One has the following more studied cases:

1° Ω and η are both closed forms. Then M is called a cosymplectic manifold.

2° $d\eta = 0$, $d\Omega = 2\eta \wedge \Omega$. Then M is called a Kenmotsu manifold.

3° $d\eta = \omega \wedge \eta$, $d\Omega = 2\omega \wedge \Omega$. Then M is called a locally conformal cosymplectic manifold (see [3],[16]). In this case ω and its dual vector $T = b^{-1}(\omega)$ with respect to g is called the Lee form (or characteristic form) and Lee vector field respectively.

In the present paper we consider an almost cosymplectic manifold $M(\Omega, \eta, \xi, g)$ carrying a globally defined vector field T whose dual form $b(T)$ is denoted by ω.

Next denote by $0 = \text{vect}\{e_A ; A = 0, 1, \ldots, 2m\}$ an orthonormal vector basis on M and by $\left\{\theta^A_B\right\}$ the associated connection forms. If the connection forms satisfy

$$\theta^A_B = \langle T, e_B \wedge e_A \rangle ; \quad \wedge \text{ is the wedge product},$$

then one has

$$\nabla_T e_A = 0$$

Therefore we agree to say that M is structured by a T-parallel connection. In this condition the following significative fact emerges: the almost cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ of M moves to an exact locally conformal cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ (abbreviated exact L.C.C.), having T (resp. $\omega = -df/f$) as Lee vector field (resp. Lee form).

Moreover any such a manifold M is a space form of curvature $-2c$ and f is the energy function corresponding to a Hamiltonian vector field associated with T (in the sense of [3]). If θ (resp. Θ) represents the indexless (or generic) connection forms (resp. curvature forms) of M, then T defines an infinitesimal homothety of θ, i.e. $L_T\theta = 2c\theta$, and a relative infinitesimal T conformal transformation of Θ and Ω, i.e.

$$d(L_T\Theta) = 2c\omega \wedge \Theta, \quad d(L_T\Omega) = 2c\omega \wedge \Omega .$$

In Section 3 the existence of a structure conformal vector field C on M is proved, i.e.

$$\nabla_Z C = \lambda Z + g(Z,T)C - g(Z,C)T ; \quad \lambda \in C^\infty M , \quad Z \in \Gamma(TM) .$$

Moreover C is a divergence conformal vector field, i.e. grad (div C) is a concurrent vector field and it defines an infinitesimal conformal transformation of:

i) the conformal cosymplectic form Ω, i.e. $L_C\Omega = \rho\Omega$, $\rho = 2\lambda$;

ii) the dual forms ω^A, i.e. $L_C\omega^A = \frac{\rho}{2}\omega^A$;

iii) the curvature forms Θ^A_B, i.e. $L_C\Theta^A_B = \rho\Theta^A_B$;

iv) all the $(2q + 1)$-forms $\alpha_q = b(C) \wedge \Omega^q$, i.e. $L_C\alpha_q = (1 + q)\rho\alpha_q$;

v) all the functions $g(C,Z)$, i.e. $L_C g(C,Z) = \rho g(C,Z), Z \in \Gamma(TM)$.

In the last section, we discuss some properties of the tangent bundle manifold TM having as basis the exact (L.C.C.)-manifold M. Denote by V, γ and ν the Liouville vector field ([13]), the Liouville 1-form and the Liouville function respectively, on TM.

The following properties are proved:

i) the complete lift Ω^c of Ω is a $d^{-\omega}$-exact 2-form (d^ω is the cohomological operator [11]) and is homogeneous of class 1, i.e.

$$L_V\Omega^c = \Omega^c ;$$

ii) γ satisfies $d^{-\omega}\gamma = \psi$ and ψ is a Finslerian form, i.e.

$$L_V\psi = \psi , \quad \dot{i}_v\psi = 0$$

($\dot{i}_v$ denotes the vertical differentiation operator [11]);

iii) the vertical lift T^v of T defines an infinitesimal automorphism of ψ, i.e. $LT^v\psi = 0$;

iv) the function $r = f\nu$ and the 2-form $f\psi$ define a regular mechanical system $\mathcal{M}$ ([13]) having r as kinetic energy and $f\psi$ as canonical symplectic (exact) form.

1. PRELIMINARIES

Let (M,g) be a Riemannian C^∞-manifold and let ∇ be the covariant differential operator with respect to the metric tensor g. Assume that M is oriented and ∇ is a Levi-Civita connection. Let $\Gamma(TM) = \chi(M)$ and $b: TM \to T^*M$ be the set of sections of the tangent bundle TM and the musical isomorphism ([18]) defined by g, respectively. Following [18] we set

$$A^q(M,TM) = \Gamma\, Hom(\Lambda^q TM, TM)$$

and notice that elements of $A^q(M,TM)$ are vector valued q-forms ($q \le dim M$).

Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$ the exterior covariant derivative operator with respect to ∇. It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$ unlike $d^2 = d \circ d = 0$. If $p \in M$, then the vector valued 1-form $dp \in A^1(M,TM)$ is the canonical vector valued 1-form of M ([5]) and since ∇ is symmetric one has $d^\nabla(dp) = 0$. The operator

$$d^\omega = d + e(\omega) \tag{1.1}$$

acting on ΛM, where $e(\omega)$ means the exterior product by the closed 1-form ω, is called the cohomological operator ([11]). One has

$$d^\omega o d^\omega = 0 . \tag{1.2}$$

Any form $u \in \Lambda M$ such that $d^\omega u = 0$ is said to be d^ω-closed and if ω is an exact form, then u is said to be a d^ω-exact form. Any vector field $Z \in \Gamma(TM)$ such that

$$d^\nabla(\nabla Z) = \nabla^2 Z = \pi \wedge dp \in A^2(M,TM) \tag{1.3}$$

for some 1-form π, is said to be an exterior concurrent vector field ([17]). The form π which is called the concurrence form is given by

$$\pi = \lambda b(Z); \quad \lambda \in C^\infty M . \tag{1.4}$$

A non flat manifold of dimension $m > 2$ is an elliptic or hyperbolic space-form if and only if every vector field on M is an exterior concurrent one ([17]). On the tangent bundle manifold TM, d_v and i_v define the vertical differentiation and the vertical derivation operators respectively ([7]). d_v is an anti-derivation of degree 1 on $\Lambda(TM)$ and i_v is a derivation of degree 0 on $\nabla(TM)$.

In an n-dimensional Riemannian manifold M, denote by

$$\mathbf{O} = vect\{e_A; A = 1, \dots, n\}$$

a local field of orthonormal frames and let

$$\mathbf{O}^* = covect\{\omega^A; A = 1, \dots, n\}$$

be its associated coframe.

The soldering form dp is expressed by

$$dp = \omega^A \otimes e_A \tag{1.5}$$

and E. Cartan's structure equations written indexless manner are

$$\nabla e = \theta \otimes e \tag{1.6}$$

$$d\omega = -\theta \wedge \omega \tag{1.7}$$

$$d\theta = -\theta \wedge \theta + \Theta \tag{1.8}$$

Any vector field T such that

$$\nabla T = s\, dp + u \otimes T , \quad u \in \Lambda^1 M \tag{1.9}$$

is called a torse forming (K. Yano [20]). If $du = 0$, then T is a closed torse forming, which implies that T is an exterior concurrent vector field, and if $u = 0$, then T is a concurrent vector field ([22]).

Let now W be any conformal vector field on M (i.e. the conformal version of Killing's equations). As is well known, W satisfies

$$L_W g = \rho g \quad \text{or} \quad g(\nabla_Z W, Z') + g(\nabla_{Z'} W, Z) = \rho g(Z, Z') \tag{1.10}$$

where the conformal scalar ρ is defined by

$$\rho = \frac{2}{n}(div W) . \tag{1.11}$$

We recall some basic formulas which we shall use in the following sections.

$$L_W b(Z) = \rho b(Z) + b[W,Z] \qquad \text{(Orsted lemma)} \tag{1.12}$$

$$L_W K = (n-1)\Delta\rho - K\rho \tag{1.13}$$

$$2L_W S(Z,Z') = (\Delta)\rho g(Z,Z') - (n-2) \quad (\mathrm{Hess}\,\nabla^{\rho})(Z,Z') . \tag{1.14}$$

In the above equations L_W, K, Δ and S denote the *Lie* derivative with respect to W, the scalar curvature of M, the Laplacian and the Ricci tensor field of ∇, respectively. One has

$$(\mathrm{Hess}_{\nabla}\rho)(Z,Z') = g(Z, H_\rho Z'), \quad H_\rho Z' = \nabla_{Z'}(\mathrm{grad}\,\rho)$$

(see also [2]).

2. EXACT LOCALLY CONFORMAL COSYMPLECTIC MANIFOLDS

Let (M,g) be a $(2m+1)$-dimensional oriented Riemannian C^∞-manifold and let $T = \sum_{A=0}^{2m} t^A e_A$ and $\omega = b(T)$ be a globally defined vector field on M and its dual form respectively.

Denote by $\mathbf{O} = vect\{e_A \,;\, A = 0, 1, \ldots, 2m\}$ (resp. θ_B^A) a local field of orthonormal frames on M (resp. the associated connection forms). Recall that the vectorial wedge product $\wedge$ is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y \,; \quad Z \in \Gamma(TM)$$

i.e.

$$X \wedge Y = b(Y) \otimes X - b(X) \otimes Y .$$

Assume now that all the connection forms θ satisfy

$$\theta_B^A = \langle T, e_B \wedge e_A \rangle . \tag{2.1}$$

Then by the structure equations (1.6), it follows at once

$$\theta_B^A = t^B \omega^A - t^A \omega^B . \tag{2.2}$$

It should be noticed that if θ satisfy (2.2) one has $\theta(T) = 0$ and the above equation shows that all the connection forms θ are relations of integral invariance for the vector field T (in the sense of A. Lichnerowicz [14]).

Next by the structure equations (1.6) and by (2.2) one obtains

$$\nabla e_A = t^A dp - \omega^A \otimes T \tag{2.3}$$

and the above equation implies

$$\nabla_T e_A = 0 . \tag{2.4}$$

From (2.4) the following significative fact emerges: all the vectors of the **O**-basis are T-parallel. Therefore we agree to say that the Riemannian manifold under consideration is structured by a T-parallel connection (abr. T.P.).

Further again by (2.2) one derives by the structure equations (1.7)

$$d\omega^A = \omega \wedge \omega^A \,; \quad \omega = b(T) = t^A \omega^A \tag{2.5}$$

which by a simple argument implies that the dual form ω of T is closed, i.e.

$$d\omega = 0 . \tag{2.6}$$

Thus in terms of d^ω-cohomology, (2.5) may be written as

$$d^{-\omega}\omega^A = 0 \tag{2.7}$$

and $\mathbf{O}^* = \{\omega^A\}$ is defined as a $d^{-\omega}$-closed covector basis.

Now for reasons which will soon appear, we set

$$\omega^0 = \eta , \quad e_0 = \xi \tag{2.8}$$

and consider on M the globally defined 2-form Ω of rank $2m$ given by

$$\Omega = \Sigma\omega^a \wedge \omega^{a^*} ; \quad a = 1,\ldots,m ; \quad a^* = a + m . \tag{2.9}$$

Then since $\Omega^m \wedge \eta \neq 0$, $i_\xi\Omega = 0$, one may say that the triple (Ω, η, ξ) defines an almost cosymplectic structure $1 \times Sp(2m, \mathbf{R})$ having ξ as Reeb's vector field.

Next taking the exterior differential of Ω a short calculation gives with the help of (2.5)

$$d\Omega = 2\omega \wedge \Omega \Leftrightarrow d^{-2\omega}\Omega = 0 \tag{2.10}$$

and by (2.5) we may write

$$d\eta = \omega \wedge \eta \Leftrightarrow d^{-\omega}\eta = 0 . \tag{2.11}$$

We conclude that any odd dimensional Riemannien manifold M structured by a T-parallel connection is endowed with a locally conformal cosymplectic structure $1 \times CSp(2n, \mathbf{R})$ (abr. L.C.C.). We notice that the vector field T (resp. the 1-form $\omega = b(T)$) is the Lee vector field (resp. the Lee form) of this structure.

Moreover since $\omega = t^A\omega^A$, then by a simple argument it follows on behalf of (2.5) that one may set

$$dt^A = f\omega^A ; \quad f \in C^\infty M \tag{2.12}$$

which by exterior differentiation gives instantly

$$\omega = -df/f . \tag{2.13}$$

Therefore since ω is an exact form, it follows on behalf of a known terminology, that the manifold M under consideration is an exact (L.C.C.)-manifold. We agree to call f the distinguished scalar field associated with the exact (L.C.C.)-structure.

Now taking the covariant differential of T one finds by (2.3) and (2.12)

$$\nabla T = (f + 2l)dp - \omega \otimes T \tag{2.14}$$

where we have set

$$g(T,T) = 2l . \tag{2.15}$$

Using (2.12) and (2.15), we have

$$dl = f\omega \Rightarrow l + f = c = const \neq 0 \tag{2.16}$$

and (2.14) becomes

$$\nabla T = (l + c)dp - \omega \otimes T . \tag{2.17}$$

Hence, by (1.9) and (2.6) T is a closed torse forming and consequently an exterior concurrent (abr. E.C.)-vector field.

Operating now on ∇e_A and ∇T by the exterior covariant derivative operator d^∇, one gets by (2.12) and (2.16)

$$d^\nabla(\nabla e_A) = \nabla^2 e_A = 2c\omega^A \wedge dp \tag{2.18}$$

$$d^\nabla(\nabla T) = \nabla^2 T = 2c\omega \wedge dp . \tag{2.19}$$

From the above equations it is seen that any vector field Z on M is E.C. with constant conformal scalar $2c$. Therefore on behalf of the general properties of E.C.-vector fields ([17]), we may state the following striking property: the exact L.C.C.-manifold $M(\Omega, \eta, \xi)$ under discussion is a space-form of curvature $-2c$.

As a consequence, it follows that the curvature forms Θ are expressed by

$$\Theta^A_B = -2c\omega^A \wedge \omega^B \tag{2.20}$$

Next taking the exterior differential of the forms Θ, one quickly finds by

$$d\Theta^A_B = 2\omega \wedge \Theta^A_B \Leftrightarrow d^{-2\omega}\Theta^A_B = 0 \tag{2.21}$$

which shows that all the curvature forms Θ are $d^{-2\omega}$-exact.

On the other hand taking the Lie derivatives of the covectors ω^A of $\mathbf{O}^*$ one derives by (2.12) and (2.16)

$$L_l\,\omega^A = (l+c)\omega^A - t^A\omega\,. \tag{2.22}$$

Therefore since L_l satisfies Leibniz rule one deduces by (2.20)

$$L_l\Theta^A_B = 2(l+c)\Theta^A_B + 2c\,\theta^A_B \wedge \omega \tag{2.23}$$

Similarly, we obtain

$$d\Theta^A_B = 2f\omega^B\Lambda\omega^A + \omega\Lambda\theta^A_B \tag{2.24}$$

Clearly by (2.12) one has $L_l t^A = ft^A$ and with the help of (2.22) we deduce

$$L_l\theta^A_B = 2c\,\theta^A_B\,. \tag{2.25}$$

Accordingly by the above equations we may say that the Lie vector field T defines on infinitesimal homothety of all the connection forms θ.

Taking now the exterior differential of the equations (2.23), a standard calculation gives

$$d(L_T\Theta^A_B) = 8c\,\omega \wedge \Theta^A_B \tag{2.26}$$

which proves that T defines a relative infinitesimal conformal transformation ([19]) of the curvature forms.

let $\mu : TM \to T^*M$, $\mu(Z) = i_Z\Omega$ be the bundle isomorphism defined by Ω and set $\overline{\omega} = \mu(T)$, i.e.

$$\overline{\omega} = i_T\Omega = \sum_{a=1}^{m}(t^a\omega^{a^*} - t^{a^*}\omega^a) \tag{2.27}$$

for the dual form of T with respect to Ω. By (2.5) and (2.12) an easy calculation gives

$$d\overline{\omega} = 2f\Omega + \omega \wedge \overline{\omega} \tag{2.28}$$

and by (2.10) and (2.13) one gets

$$L_T\Omega = 2(l+c)\Omega + \overline{\omega} \wedge \omega \tag{2.29}$$

and consequently by (2.28) it follows

$$d(L_T\Omega) = 2c\,\omega \wedge \Omega\,. \tag{2.30}$$

Hence as for the curvature forms Θ, T defines a relative conformal transformation of the structure 2-form Ω.

Consider now the vector valued 1-form

$$F = \omega^a \otimes e_{a^*} - \omega^{a^*} \otimes e_a \in A^1(M,TM)\,. \tag{2.31}$$

If Z is any vector field, a simple calculation gives

$$\langle F,Z\rangle = Z^a e_{a^*} - Z^{a^*} e_a = \overline{Z} \tag{2.32}$$

which implies

$$g(Z,Z') + g(Z,\overline{Z}') = 0\,, \quad Z,Z' \in \Gamma(TM) \tag{2.33}$$

and $\langle F,dp\rangle = 2\Omega$.

On the other hand since $\overline{\omega}(T) = 0$ one gets by (2.27)

$$L_T\overline{\omega} = 2c\overline{\omega} \tag{2.34}$$

that is T defines an infinitesimal homothety of $\overline{\omega} = (\mu \circ b)T$.

Next by (2.12) and (2.13) one easily gets

$$i_{\bar{T}}\Omega = \frac{df}{f} - \frac{1}{f}\xi(f)\eta \,. \tag{2.35}$$

Therefore by reference to [3] one may call $\overline{T}$ the cosymplectic Hamiltonian vector field of M and the distinguished scalar f turns out to be the energy function corresponding to $\overline{T}$.

Moreover by (2.35) one derives

$$L_{\bar{T}}\Omega = \eta(\overline{T})\eta \wedge \omega \Rightarrow d(L_{\bar{T}}\Omega) = 0 \tag{2.36}$$

which shows that $\overline{T}$ defines a relative infinitesimal automorphism (R. Abraham [1]) of Ω.

Summing up, we state the following

THEOREM. Let M be a $(2m+1)$-dimensional Riemannian manifold and let T be a globally defined vector field on M. If M is structured by a T-parallel connection, then M is endowed with an exact locally conformal cosymplectic structure $1 \times CSp(2m, \mathbf{R})$, having T (resp. $\omega = b(T)$) as Lee vector (resp. Lee form) and any such an M is a space-form of curvature $-2c$.

Moreover one has the following properties:

i) T defines an infinitesimal homothety of the connection forms θ and of the 1-form $\mu(T)$, i.e.

$$L_T\theta = 2c\theta \,, \quad L_T\mu(T) = 2c\mu(T)$$

ii) T defines a relative infinitesimal conformal transformation of the curvature forms Θ and of the structure 2-form Ω, i.e.

$$d(L_T\Theta) = 8c\omega \wedge \Theta \,, \quad d(L_T\Omega) = 2c\omega \wedge \Omega$$

iii) the vector field $\overline{T} = (b^{-1} \circ \mu)\, T$ (resp. f) is the cosymplectic Hamiltonian associated with the $1 \times CSp(2m, \mathbf{R})$-structure of M (resp. its corresponding energy function) and $\overline{T}$ defines a relative infinitesimal automorphism of Ω.

Let now $\Phi : M \to \tilde{M}$ be a conformal diffeomorphism (abr. C.D.) that is

$$\Phi : g \to e^{2\sigma} g = \tilde{g} \,; \quad \sigma \in C^{\infty}M \,.$$

One also say that g and $\tilde{g}$ are conformally equivalent metrics and setting $e^{2\sigma} = \nu^2$, we agree to call the function ν the argument of the C.D.

As is shown one has for $Z, Z' \in \Gamma(TM)$

$$\tilde{\nabla} Z = \nabla Z + b(\operatorname{grad}\sigma) \otimes Z - b(Z) \otimes \operatorname{grad}\sigma + g(Z, \operatorname{grad}\sigma) dp \tag{2.37}$$

or equivalently

$$\tilde{\nabla}_{Z'} Z = \nabla_{Z'} Z + Z'(\sigma) Z + Z(\sigma) Z' - g(Z, Z')\operatorname{grad}\sigma \tag{2.38}$$

and if K and $\tilde{K}$ denote the scalar curvature of M and $\tilde{M}$ respectively then one has ([8])

$$\tilde{K} = e^{-2\sigma}\{K + 2(n-1)(n-2)\|\operatorname{grad}\sigma\|^2\} \tag{2.39}$$

($n = dim M$).

If M is an exact (L.C.C.)--manifold, its Ricci tensor field S satisfies

$$S(Z, Z') = -4mc\, g(Z, Z') \,; \quad Z, Z' \in \Gamma(TM) \tag{2.40}$$

and the scalar curvature K is given by

$$K = -4m(2m+1)c \,. \tag{2.41}$$

Perform now a conformal transformation of M having as argument e^{σ} the energy function f. It is obvious that

$$(2.42) d\sigma = df/f = -\omega \,. \tag{2.42}$$

Then we have $\operatorname{grad}\sigma = -T$, which implies

$$\Delta\sigma = div\, T = (2m+1)c + (2m-1)l\,. \tag{2.43}$$

Hence by (2.41) and (2.43) we derive at once from (2.39), $\tilde{K} = 0$, that is $\tilde{M}$ is a flat manifold. We notice that this fact is in accordance with the known

PROPOSITION. A Riemannian manifold of constant curvature is conformally flat, provided $n \geq 3$.

Using (2.37) one may prove that all vectors $\tilde{e}_A$ are parallel (the connection forms $\tilde{\theta}^A_B$ vanish, i.e. $\tilde{\nabla}$ is a flat connection). Thus we have

PROPOSITION. If M is an exact (L.C.C.)-manifold with metric tensor g and energy function f, then the metric $f^2 g$ is flat.

3. STRUCTURE CONFORMAL VECTOR FIELDS ON AN EXACT (L.C.C.)-MANIFOLD

In consequence of some conformal properties induced by the T-parallel connection which structures $M(\Omega,\eta,\xi,g)$ we are naturally led to see if the manifold M under consideration carries a structure conformal vector field C in the sense of [6], [15]. Therefore the covariant differential of C is expressed by

$$\nabla C = \lambda dp + C \wedge T = \lambda dp + \omega \otimes C - \alpha \otimes T\,; \quad \lambda \in C^\infty M\,, \quad \alpha = b(C)\,. \tag{3.1}$$

Put

$$C = C^A e_A \Rightarrow b(C) = \alpha = C^A \omega^A \tag{3.2}$$

and $s = g(C,T)$. Then by (2.3) and (3.1) one quickly gets

$$dC^A = (\lambda - s)\omega^A + C^A \omega \tag{3.3}$$

$$d\alpha = 2\omega \wedge \alpha \Rightarrow d^{-2\omega}\alpha = 0\,. \tag{3.4}$$

Next since $ds = \langle \nabla C, T\rangle + \langle \nabla T, C\rangle$,a short calculation gives

$$ds = \lambda\omega - (l-c)\alpha \tag{3.5}$$

$$ds = d\lambda \tag{3.6}$$

By (3.4), (3.5) and (3.6) it is seen that the existence of C is assured by an exterior differential system Σ whose characteristic numbers are

$$r = 3\,, \quad s_0 = 2\,, \quad s_1 = 1\,.$$

Then Σ is in involution in the sense of E. Cartan (i.e. $r = s_c + s_1$). Accordingly one may say that the existence of C depends on 2 arbitrary functions of one argument (E. Cartan's test). The conformal scalar ρ associated with $C(L_C g = \rho g)$ is given by

$$\rho = 2\lambda\,. \tag{3.7}$$

By a short calculation one has

$$[C,T] = -\lambda T - (l-c)C\,; \quad [\quad]\text{: Lie bracket} \tag{3.8}$$

and from (3.5) it follows

$$L_C\omega = ds = \lambda\omega - (l-c)\alpha\,. \tag{3.9}$$

This equation matches by Orsted's lemma (1.12) the expression of $[C,T]$.

On the other hand since C is necessarily an E. C. vector field (M is a space-form), then operating (3.1) by d^∇ and taking account of (3.4) and (3.5), one derives

$$d^\nabla(\nabla C) = \nabla^2 C = 2c\alpha \wedge dp\,. \tag{3.10}$$

The above equation is coherent with the properties obtained in Section 2.

Setting now

$$\bar{\alpha} = \iota_C\Omega = \Sigma(C^a\omega^{a^*} - C^{a^*}\omega^a) \tag{3.11}$$

one gets by (3.4) and (2.5)

$$d\bar{\alpha} = 2(\lambda - s)\Omega + 2\omega \wedge \bar{\alpha} \tag{3.12}$$

and one follows

$$L_C \Omega = \rho\Omega . \tag{3.13}$$

Hence (3.13) reveals that C defines an infinitesimal conformal transformation (abr. I.C.T.) of the conformal cosymplectic form Ω.

By similar methods, one gets by (2.5), (2.24), (2.20) and (2.21)

$$L_C\omega^A = \frac{\rho}{2}\omega^A , \quad L_C\theta^A_B = \frac{\rho}{2}\theta^A_B , \quad L_C\Theta^A_B = \rho\Theta^A_B . \tag{3.14}$$

Therefore one may say that C defines an I.C.T. of the exact (L.C.C.)-structure of M.

Moreover let L be the operator of type (1.1) on forms defined by S. Goldberg ([8]), that is $Lu = u \wedge \Omega; u \in \Lambda^1 M$, and consider on M the $(2q+1)$-forms

$$L^q\alpha = \alpha_q = \alpha \wedge \Omega^q . \tag{3.15}$$

Since by Orsted's lemma one has

$$L_C\alpha = \rho\,\alpha \tag{3.16}$$

then by (3.13) and a standard calculation one derives

$$L_C\alpha_q = (q+1)\rho\,\alpha_q . \tag{3.17}$$

Hence C defines an (I.C.T.) of all the $(2q+1)$-forms α_q.

Next since C is a conformal vector field, then as is known (see (1.11)) one has

$$div\ C = (\rho/2)(2m+1) \tag{3.18}$$

and since $\rho = 2\lambda$ it follows by (3.5) and (3.6) that

$$\text{grad}\,\rho = \rho T + 2(c - l)C . \tag{3.19}$$

Further by (2.16) and taking account of (2.14) and (3.1) it is easily deduced

$$\nabla\,\text{grad}\,\rho = 2c\rho dp . \tag{3.20}$$

Thus one may state the following relevant property: the gradient of the associated scalar ρ of C is a concurrent vector field (K. Yano and B. Y. Chen [22]). We agree to call a conformal vector field such that the gradient of its conformal scalar ρ is a concurrent vector field, a divergence conformal vector field. Such a situation occurs also when studying conformal vector fields on Lorentzian P.S. manifolds (see I. Mihai and R. Rosca [15]).

On the other hand from (2.14) one derives

$$div\ T = (2m-1)l + (2m+1)c \tag{3.21}$$

and since $div\ C = (2m+1)\lambda$, one gets on behalf of (3.20)

$$\Delta\rho = -div(\text{grad}\,\rho) = -2(2m+1)c\rho \tag{3.22}$$

which shows that ρ is an eigenfunction of Δ.

C being an E.C. vector field satisfying (3.10), one has ([17])

$$S(C,Z) = -4mc\ g(C,Z) , \quad Z \in \Gamma(TM) \tag{3.23}$$

where S denotes the Ricci tensor field of ∇.

Now making use of (1.14) and carrying out the calculations, one finds by (3.19) and (3.22)

$$L_C g(C,Z) = \rho g(C,Z) . \tag{3.24}$$

Hence the vector field C defines an I.C.T. of all the functions $g(C,Z)$, where $Z \in \Gamma(TM)$.

Concuding, we have proved the following

THEOREM. Let M be the exact (L.C.C.) manifold defined in Section 2 and C a structure conformal vector field on M (which existence is proved), i.e.

$$\nabla C = \frac{\rho}{2} dp + C \wedge T\,; \quad L_C g = \rho g$$

Then C is a divergence conformal vector field (i.e. grad($div\ C$) is a concurrent vector field) and it defines the following infinitesimal conformal transformations

$$L_C\Omega = \rho\Omega\,, \quad L_C\omega^A = \frac{\rho}{2}\omega^A\,, \quad L_C\theta^A_B = \frac{\rho}{2}\theta^A_B$$

$$L_C\Theta^A_B = \rho\Theta^A_B\,, \quad L_C\alpha_q = (1+q)\rho\alpha_q\,, \quad L_C g(C,Z) = \rho g(C,Z)(Z \in \Gamma(TM))$$

where Ω, ω^A, θ^A_B, Θ^A_B and $\alpha_q = b(C) \wedge \Omega^q$ are the conformal symplectic 2-form, the dual forms, the connection forms, the curvature forms and the $(2q+1)$-forms defined by the (1,1)-operator L, respectively on M.

4. GEOMETRY OF THE TANGENT BUNDLE OF AN EXACT (L.C.C.)-MANIFOLD

Let now TM be the tangent bundle manifold having the exact (L.C.C.)-manifold M discussed in Section 2 as a basis.

Denote by $V(v^A)(A = 0,1,\ldots,2m)$ the Liouville vector field (or the canonical vector field [7]). Accordingly we may consider the set $B^* = \{\omega^A, dv^A\}$ as an adapted cobasis in TM. Following Godbillon ([7]) we denote by d_v and i_v the vertical differentiation and the vertical derivative operators with respect to B^*, respectively (d_v is an antiderivation of degree 1 on $\Lambda(TM)$ and i_v is a derivation of degree 0 on $\Lambda(TM)$). Let $T^r_s M$ be the set of all tensor fields of type (r,s) on M.

In general as is known ([23]) the vertical and complete lifts are linear mappings of $T^r_s M$ into $T^r_s(TM)$ and one has

$$(T_1 \otimes T_2)^c = T_1^v \otimes T_2^c + T_1^c \otimes T_2^v\,. \tag{4.1}$$

In the case under discussion we may define the complete lift Ω^c of the structure 2-form Ω of M by the 2-form of rank $4m$ on TM

$$\Omega^c = \Sigma(dv^a \wedge \omega^{a^*} + \omega^a \wedge dv^{a^*})\,, \quad a = 1,\ldots,m\,; \quad a^* = a+m\,. \tag{4.2}$$

On the other hand since the Liouville vector field V is expressed by

$$V = \Sigma\, v^A \frac{\partial}{\partial v^A} \tag{4.3}$$

then as is known the basic 1-form

$$\gamma = \Sigma\, v^A \omega^A \tag{4.4}$$

is called the Liouville form (see also [13]).

Taking now the exterior differential of Ω^c one finds by (2.5)

$$d\Omega^c = \omega \wedge \Omega^c \Leftrightarrow d^{-\omega}\Omega^c = 0 \tag{4.5}$$

which shows that Ω^c is similarly as Ω a $d^{-\omega}$-exact form. We recall that in general conformal properties are not preserved by complete lifts ([23]).

One has

$$i_V\Omega^c = \Sigma(v^a\omega^{a^*} - v^{a^*}\omega^a) \tag{4.6}$$

which implies $\omega(V) = 0$ and so by (4.5) and (4.6) one gets

$$L_V\Omega^c = \Omega^c\,. \tag{4.7}$$

Accordingly on behalf of a known definition ([13]), the above equation shows that Ω^c is of class 1, a homogeneous form on TM. Taking now the exterior differential of the Liouville form γ defined by (4.4), one gets at once by (2.5)

$$d\gamma = \omega \wedge \gamma + \psi \Leftrightarrow d^{-\omega}\gamma = \psi \tag{4.8}$$

where we have set

$$\psi = \Sigma\, dv^A \wedge \omega^A . \tag{4.9}$$

From (4.8) and (1.2) one obtains instantly

$$d^{-\omega}\psi = 0 \Leftrightarrow d\psi = \omega \wedge \psi . \tag{4.10}$$

Since clearly the 2-form ψ is of maximal rank, we agree to call ψ the canonical conformal symplectic form of M. Noticing that one has

$$i_V\psi = \gamma , \quad \omega(V) = 0 \tag{4.11}$$

which implies

$$L_V\psi = \psi . \tag{4.12}$$

Hence ψ is as Ω^c a homogeneous of class 1, 2-form.

Next making use of the vertical operator i_v defined by $i_v\lambda = 0$, $i_v dv^A = \omega^A$, $i_v\omega^A = 0(\lambda \in C^\infty M)$ one quickly finds by (4.9)

$$i_v\psi = 0 \tag{4.13}$$

and the above equation together with (4.12) proves that ψ is a Finslerian form ([7]).

We recall that the vertical lift Z^v ([23]) of a vector field $Z \in \Gamma(TM)$ with components Z^A in M, has as components

$$Z^v = \begin{pmatrix} 0 \\ Z^A \end{pmatrix} = Z^A \frac{\partial}{\partial v^A} .$$

Hence in the case under consideration one has

$$T^v = \Sigma\, t^A \frac{\partial}{\partial v^A}; \quad A = 0,1,\ldots,2m \tag{4.14}$$

and by (4.9) one gets

$$i_{T^v}\psi = \omega . \tag{4.15}$$

Therefore by (4.10) one derives

$$L_{T^v}\psi = 0 \tag{4.16}$$

and one may say that T^v defines an infinitesimal automorphism of ψ.

Finally we set

$$r = fv \tag{4.17}$$

where

$$v = \frac{1}{2}\Sigma(v^A)^2 \tag{4.18}$$

denotes the Liouville function on M ([9]).

Operating on r by the vertical differentiation operator d_v ([7]) one gets

$$d_v r = f\sum_A v^A\omega^A = f\mu \tag{4.19}$$

and taking the exterior differential of (4.19) we obtain by (2.13) and (4.9)

$$d(d_v r) = f\,\Sigma\, dv^A \wedge \omega^A = f\psi . \tag{4.20}$$

Next putting $II = f\psi$ it follows by (2.13)

$$dII = 0 . \tag{2.21}$$

Therefore the exact symplectic form II can be viewed as the canonical symplectic form of the $(4m+2)$-dimensional manifold TM ([13]).

Finally by reference to [13] one may consider that the pair (r, II) defines a regular mechanical system $\mathcal{M}$ (in the sense of Klein [13]) having the scalar r as kinetic energy.

THEOREM. Let TM be the tangent bundle manifold having as basis the exact (L.C.C.)-manifold $M(\Omega, T, \omega)$ discussed in Section 2. Let V, γ and v be the Liouville vector field, the Liouville form and the Liouville function of TM, respectively. One has the following properties:

i) the complete lift Ω^c on TM of the conformal cosymplectic form Ω of M is a homogeneous of class 1, 2-form, i.e. $L_V \Omega^c = \Omega^c$, and it is $d^{-\omega}$-exact, i.e. $d^{-\omega} = 0$;

ii) γ satisfies $d^{-\omega}\gamma = \psi \Rightarrow d^{-\omega}\psi = 0$ and ψ is the canonical conformal symplectic form of TM and ψ enjoys also the property to be a Finslerian form;

iii) the vertical lift T^v of T defines an infinitesimal automorphism of ψ, i.e. $L_{T^v}\psi = 0$;

iv) $r = fv$ and $f\psi$ define a regular mechanical system on TM having r as kinetic energy and $f\psi$ as canonical symplectic form (where f is the energy function of M).

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