ABSTRACT. Fixed point theorems are given for non-self maps and pairs of non-self maps defined on d-complete topological spaces.

KEY WORDS AND PHRASES. d-complete topological spaces, fixed points, non-self maps, pairs of mappings.

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1. INTRODUCTION.

Let \((X, t)\) be a topological space and \(d: X \times X \to [0, \infty)\) such that \(d(x, y) = 0\) if and only if \(x = y\). \(X\) is said to be d-complete if \(\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty\) implies that the sequence \(\{x_n\}_{n=1}^{\infty}\) is convergent in \((X, t)\). Complete metric spaces and complete quasi-metric spaces are examples of d-complete topological spaces. The d-complete semi-metric spaces form an important class of examples of d-complete topological spaces.

Let \(X\) be an infinite set and \(t\) any \(T_1\) non-discrete first countable topology for \(X\). There exists a complete metric \(d\) for \(X\) such that \(t \leq t_d\) and the metric topology \(t_d\) is non-discrete. Now \((X, t, d)\) is d-complete since \(\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty\) implies that \(\{x_n\}_{n=1}^{\infty}\) is Cauchy in \(t_d\). Thus, \(x_n \to x\), as \(n \to \infty\), in \(t_d\) and therefore in the topology \(t\). The construction of \(t_d\) is given by T. L. Hicks and W. R. Crisler in [1].

Recently, T. L. Hicks in [2] and T. L. Hicks ad B. E. Rhoades in [3] and [4] proved several metric space fixed point theorems in d-complete topological spaces. We shall prove additional theorems in this setting.

Let \(T: X \to X\) be a mapping. \(T\) is \(\omega\)-continuous at \(x\) if \(x_n \to x\) implies \(T x_n \to T x\) as \(n \to \infty\). A real-valued function \(G: X \to [0, \infty)\) is lower semi-continuous if and only if \(\{x_n\}_{n=0}^{\infty} = 0\) is a sequence in \(X\) and \(\lim_{n \to \infty} G(x_n) = p\) implies \(G(p) \leq \liminf_{n \to \infty} G(x_n)\).

2. RESULTS.

In [2], Hicks gave the following result.

THEOREM ([2], Theorem 2): Suppose \(X\) is a d-complete Hausdorff topological space, \(T: X \to X\) is \(\omega\)-continuous and satisfies \(d(T x, T^2 x) \leq k(d(x, T x))\) for all \(x \in X\), where \(k: [0, \infty) \to [0, \infty), k(0) = 0\), and \(k\) is non-decreasing. Then \(T\) has a fixed point if and only if
there exists $x$ in $X$ with $\sum_{n=1}^{\infty} k^n(d(x, Tx)) < \infty$. In this case, $x_n = T^n x \to p \equiv Tp$. [$k$ is not assumed to be continuous and $k^2(a) = k(k(a))$.]

The following conditions are examined. Let $T : C \to X$ with $C$ a closed subset of the d-complete topological space $X$ and $C \subseteq T(C)$. Let $k : [0, \infty) \to [0, \infty)$ be such that $k(0) = 0$, $k$ is non-decreasing, and

$$ k(d(Tx, Ty)) \geq d(x, y) \quad (2.1) $$

for all $x, y \in C$, or

$$ d(Tx, Ty) \geq k(d(x, y)) \quad (2.2) $$

for all $x, y \in C$, or

$$ d(x, y) \geq k(d(Tx, Ty)) \quad (2.3) $$

for all $x, y \in C$, or

$$ k(d(x, y)) \geq d(Tx, Ty) \quad (2.4) $$

for all $x, y \in C$.

It will be shown that condition (2.1) leads to a fixed point, but that the other three conditions do not guarantee a fixed point.

**THEOREM 1.** Suppose $X$ is a d-complete Hausdorff topological space, $C$ is a closed subset of $X$, and $T : C \to X$ is an open mapping with $C \subseteq T(C)$ which satisfies $d(x, y) \leq k(d(Tx, Ty))$ for all $x, y \in C$ where $k : [0, \infty) \to [0, \infty)$, $k(0) = 0$, and $k$ is non-decreasing. Then $T$ has a fixed point if and only if there exists $x_0 \in C$ with $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$.

**PROOF.** Notice that the condition $d(x, y) \leq k(d(Tx, Ty))$ forces $T$ to be one-to-one. Hence $T^{-1}$ exists. Also, $T$ is open implies that $T^{-1}$ is continuous, and thus $\omega$-continuous.

If $p = Tp$ then $\sum_{n=1}^{\infty} k^n(d(Tp, p)) = 0 < \infty$.

Suppose there exists $x_0 \in C$ such that $\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty$. We know that $T^{-1}$ exists, so let $T_1$ be $T^{-1}$ restricted to $C$. Then $T_1 : C \to C$ and $d(T_1x, T_1y) \leq k(d(x, y))$ for all $x, y \in C$. Let $y = T_1x$. Then $d(T_1x, T_1^2x) = k(d(x, T_1x))$ for all $x \in C$. In particular, $d(T_1x_0, T_1^2x_0) \leq k(d(x_0, T_1x_0)) \leq k^2(d(Tx_0, x_0))$. By induction, $d(T_1^n x_0, T_1^nx_0) \leq k^n(d(Tx_0, x_0))$. Thus,

$$ \sum_{k=1}^{\infty} d(T_1^n x_0, T_1^nx_0) \leq \sum_{k=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty. $$

Since $X$ is d-complete, $T_1^n x_0$ converges, say to $p$. Note that $p$ is in $C$ since $C$ is closed. Now $T_1(T_1^n x_0) \to T_1p$ as $n \to \infty$ since $T_1$ is $\omega$-continuous. But $T_1^n + 1 x_0 \to p$ as $n \to \infty$, and since limits are unique in $X$, $T_1 p = p$. Now $T(T_1 p) = T(p)$ and $T(T_1p) = p$ so $Tp = p$ and $T$ has a fixed point.

**COROLLARY 1.** Suppose $T : C \to X$ where $C$ is a closed subset of a d-complete Hausdorff symmetrizable topological space with $C \subseteq T(C)$. Suppose $d(x, y) \leq [d(Tx, Ty)]^p$ where $p > 1$ for all $x, y \in C$. If there exists $x_0 \in C$ such that $d(Tx_0, x_0) < 1$, then $T$ has a fixed point.

**PROOF.** If $x \neq y$, $0 < d(x, y) \leq [d(Tx, Ty)]^p$ and $Tx \neq Ty$. Thus $T$ is one-to-one and $T^{-1}$ exists. Now $d(T^{-1}x, T^{-1}y) \leq [d(x, y)]^p$ implies that $T^{-1}$ is continuous. Hence $T$ must be
open. Let \( x_0 \) be a point in \( C \) such that \( d(Tx_0, x_0) < 1 \). If \( d(Tx_0, x_0) = 0 \), then \( x_0 \) is a fixed point of \( T \). Suppose \( 0 < d(Tx_0, x_0) < 1 \). Let \( k(t) = t^p \), and \( t = d(Tx_0, x_0) \). Note that \((at)^p < at^p\) if \( 0 < a < 1 \). Since \( t^p < t \), there is an \( a_1 \in (0, 1) \) such that \( t^p = a_1 t \). Now \((t^p)^p < t^p\) and there is an \( a_2 \in (0, 1) \) such that \( t^{2p} = a_2 t^p \). But \( a_2 t^p = t^p = (a_1 t)^p = a_1 t^p \). Hence \( a_2 < a_1 \).

Now \( t^{(n+1)p} = (t^p)^p = (a_1 t^p)^p = a_1 t^{np} = a_1 t^p \). Hence, by induction, \( t^{np} < a_1^p t \) for all natural numbers \( n \). Therefore,

\[
\sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) = \sum_{n=1}^{\infty} [d(Tx_0, x_0)]^{np} = \sum_{n=1}^{\infty} t^{np} < \sum_{n=1}^{\infty} a_1^p t < \infty
\]

since \( 0 < a_1 < 1 \). Applying Theorem 1, we get that \( T \) has a fixed point.

If \( T \) is not open one could check the following condition.

**THEOREM 2.** Let \( X \) be a \( d \)-complete Hausdorff topological space, \( C \) be a closed subset of \( X \), \( T : C \to X \) with \( C \subseteq T(C) \). Suppose there exists \( k : [0, \infty) \to [0, \infty) \) such that \( k(d(Tx, Ty)) \geq d(x, y) \) for all \( x, y \in C \), \( k \) is non-decreasing, \( k(0) = 0 \), and there exists \( x_0 \in C \) such that \( \sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty \). If \( G(x) = d(Tx, x) \) is lower semi-continuous on \( C \) then \( T \) has a fixed point.

**PROOF.** If \( x \neq y \), \( 0 < d(x, y) \leq k(d(Tx, Ty)) \) so that \( d(Tx, Ty) \neq 0 \). Hence \( T \) is one-to-one and \( T^{-1} \) exists. Let \( T_1 \) be \( T^{-1} \) restricted to \( C \). Now \( T_1 : C \to C \) and for \( x, y \in C \),

\[
d(x, T_1 x) \leq k(d(Tx, x)) \quad \text{and} \quad d(T_1 x, T_1 T_1 x) \leq k(d(x, T_1 x)) \leq k^2(d(Tx, x)).
\]

By induction,

\[
d(T_1^n x, T_1^n T_1 x) \leq k^n(d(Tx, x)).
\]

There exists \( x_0 \in C \) with \( \sum_{n=1}^{\infty} k^n(d(Tx_0, x_0)) < \infty \) implies \( \sum_{n=1}^{\infty} d(T_1^n x_0, T_1^n T_1 x_0) < \infty \). Since \( X \) is \( d \)-complete there exists \( p \in X \) such that \( T_1^n x_0 \to p \) as \( n \to \infty \). Note that \( p \in C \) since \( T_1^n x_0 \in C \) for all \( n \) and \( C \) is closed. Now \( G(x) = d(Tx, x) \) is lower semi-continuous on \( C \) gives \( G(p) \leq \lim \inf G(T_1^n x_0) \) or \( d(Tp, p) \leq \lim \inf d(T_1^n x_0, T_1^n x_0) = 0 \).

Thus \( Tp = p \).

In [5], Hicks gives several examples of functions \( k \) which satisfy the condition of theorem 1 of that paper. These examples, with a slight modification, carry over to the non-self map case. The non-self map version of Example 1 is given for completeness. The other examples carry over in a similar manner.

**EXAMPLE 1.** Suppose \( 0 < \lambda < 1 \). Let \( k(t) = \lambda t \) for \( t \geq 0 \). If \( d(x, y) \leq \lambda d(Tx, Ty) \), \( T \) is open since \( T^{-1} \) exists and is continuous. Let \( x \in C \). There exists \( y \in C \) such that \( Ty = x \). Now \( d(x, y) = d(Ty, y) \leq \lambda d(T^2 y, Ty) \) and \( \sum_{n=1}^{\infty} k^n(d(Ty, y)) \leq \sum_{n=1}^{\infty} \lambda^n d(T^2 y, Ty) < \infty \). Applying Theorem 1 we get a fixed point for \( T \). (Note: \( d(x, y) \leq \lambda d(Tx, Ty) \) for \( 0 < \lambda < 1 \) is equivalent to \( d(Tx, Ty) \geq \alpha d(x, y) \) for \( \alpha > 1 \)).

The following examples show that conditions (2.2), (2.3) and (2.4) do not guarantee fixed points.

**EXAMPLE 2.** Let \( \mathbb{R} \) denote the real numbers and \( CB(\mathbb{R}, \mathbb{R}) \) denote the collection of all bounded and continuous functions which map \( \mathbb{R} \) into \( \mathbb{R} \). Let

\[
C = \{ f \in \text{CB}(\mathbb{R}, \mathbb{R}) : f(t) = 0 \text{ for all } t < 0 \text{ and } \lim_{t \to -\infty} f(t) \geq 1 \}.
\]
Define $T : C \to \text{CB}([0, \infty))$ by $Tf(t) = \frac{1}{2}f(t+1)$ and let $k(t) = \frac{1}{3}$. Then
\[ d(Tf, Tg) = \frac{1}{2}d(f, g) \geq k(d(f, g)). \]
$k$ satisfies condition (2.2) but, as shown in [6], $T$ does not have a fixed point.

**EXAMPLE 3.** Let $T : [1, \infty) \to [0, \infty)$ be defined by $Tx = x - \frac{1}{x}$ and let $k(t) = \frac{1}{2}$. Then
\[ d(Tx, Ty) \leq 2d(x, y) \text{ or } d(x, y) \geq k(d(Tx, Ty)). \]
k satisfies condition (2.3) but $T$ does not have a fixed point.

**EXAMPLE 4.** Let $c_0$ denote the collection of all sequences that converge to zero. Let $C = \{x \in c_0 : \|x\| = 1 \text{ and } x_0 = 1\}$. Define $T : C \to c_0$ by $Tx = y$ where $y_n = x_{n+1}$, $n = 0, 1, 2, \ldots$, and let $k(t) = 2t$. Then $d(Tx, Ty) = d(x, y) \leq 2d(x, y) = k(d(x, y))$ for all $x, y \in C$. $k$ satisfies condition (2.4) but, as shown in [6], $T$ does not have a fixed point.

The following theorems were motivated by the work of Hicks and Rhoades [3].

**THEOREM 3.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, and $d(Tx, T^2x) > d(x, Tx)$ for all $x \in T^{-1}(C)$ with $x \neq Tx$. Then $T$ has a fixed point in $C$.

**PROOF.** $C$ is a compact subset of a Hausdorff space so it is closed. $T$ is continuous so $T^{-1}(C)$ is closed and hence is compact since $T^{-1}(C) \subset C$. $G(x)$ is continuous so it attains its minimum on $T^{-1}(C)$, say at $z$. Let $z \in C \subset T(C)$ so there exists $y \in T^{-1}(C)$ such that $Ty = z$. If $y \neq z$ then $d(z, Ty) = d(Ty, T^2y) > d(y, Ty)$, a contradiction. Thus $y = z = Ty$ is a fixed point of $T$.

**THEOREM 4.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \leq \lambda f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies $T$ has a fixed point where $0 < \lambda < 1$, then $d(Tx, T^2x) < f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.

**PROOF.** $C$ is a compact subset of a Hausdorff space so it is closed. $T$ is continuous gives that $T^{-1}(C)$ is closed, and $T^{-1}(C) \subset C$ so $T^{-1}(C)$ is compact. Suppose $x \neq Tx$ for all $x \in T^{-1}(C)$. Then $d(x, Tx) > 0$ so that $f(d(x, Tx)) > 0$ for all $x \in T^{-1}(C)$. Define $P(x)$ on $T^{-1}(C)$ by $P(x) = \frac{d(Tx, T^2x)}{f(d(x, Tx))}$. $P$ is continuous since $T$, $f$ and $G(x)$ are continuous. Therefore $P$ attains its maximum on $T^{-1}(C)$, say at $z$. $P(z) \leq P(z) < 1$ so $d(Tx, T^2x) \leq P(z)f(d(x, Tx))$ and $T$ must have a fixed point.

**THEOREM 5.** Let $C$ be a compact subset of a Hausdorff topological space $(X, t)$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$. Suppose $T : C \to X$ with $C \subset T(C)$, $T$ and $G(x) = d(x, Tx)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $d(Tx, T^2x) \geq \lambda f(d(x, Tx))$ for all $x \in T^{-1}(C)$ implies $T$ has a fixed point where $\lambda > 1$, then $d(Tx, T^2x) > f(d(x, Tx))$ for all $x \in T^{-1}(C)$ such that $f(d(x, Tx)) \neq 0$ gives a fixed point.
PROOF. C is a compact subset of a Hausdorff space so it is closed. T is continuous gives that $T^{-1}(C)$ is closed and hence compact, since $T^{-1}(C) \subset C$. Suppose $x \neq Tx$ for all $x \in T^{-1}(C)$. Then $d(x, Tx) > 0$ and $f(d(x, Tx)) > 0$. Define $P(x) = \frac{d(Tx, T^2x)}{f(d(x, Tx))}$ so $P$ is continuous since $T$, $f$, and $G$ are continuous. $P$ attains its minimum on $T^{-1}(C)$, say at $x$. Then $P(x) \geq P(z) > 1$ so $d(Tx, T^2x) \geq P(x)f(d(x, Tx))$ and $T$ must have a fixed point.

Theorems 6, 7 and 8 are generalizations of theorems by Kang [7]. The following family of real functions was originally introduced by M. A. Khan, M. S. Khan, and S. Sessa in [8]. Let $\Phi$ denote the family of all real functions $\phi : (\mathbb{R}^+)^3 \to \mathbb{R}^+$ satisfying the following conditions:

$$(C_1) \phi \text{ is lower-semicontinuous in each coordinate variable},$$

$$(C_2) \phi(v, w) \in \mathbb{R}^+ \text{ be such that either } v \geq \phi(v, w, w) \text{ or } v \geq \phi(w, v, w). \text{ Then } v \geq hw, \text{ where } \phi(1, 1, 1) = h > 1.$$

**THEOREM 6.** Let $(X, t, d)$ be a $d$-complete topological space where $d$ is a continuous symmetric. Let $A$ and $B$ map $C$, a closed subset of $X$, into (onto) $X$ such that $C \subset A(C)$, $C \subset B(C)$, and $d(Ax, By) \geq \phi(d(Ax, x), d(By, y), d(x, y))$ for all $x, y$ in $C$ where $\phi \in \Phi$. Then $A$ and $B$ have a common fixed point in $C$.

PROOF. Fix $x_0 \in C$. Since $C \subset A(C)$ there exists $x_1 \in C$ such that $Ax_1 = x_0$. Now $C \subset B(C)$ so there exists $x_2 \in C$ such that $Bx_2 = x_1$. Build the sequence $\{x_n\}_{n=0}^{\infty}$ by $Ax_{2n+1} = x_{2n}$, $Bx_{2n+2} = x_{2n+1}$. Now if $x_{2n+1} = x_{2n}$ for some $n$, then $x_{2n+1}$ is a fixed point of $A$. Then

$$d(x_{2n+1}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$$

$$= d(Ax_{2n+1}, Bx_{2n+2})$$

$$\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}))$$

$$= \phi(0, d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})).$$

By property $(C_2)$, $d(x_{2n}, x_{2n+1}) \geq h d(x_{2n+1}, x_{2n+2})$. Hence, $x_{2n+1} = x_{2n+2}$ and $Bx_{2n+1} = Bx_{2n+2} = x_{2n+1}$. Therefore $x_{2n+1}$ is a common fixed point of $A$ and $B$. Now if $x_{2n+1} = x_{2n+2}$ for some $n$, then $Bx_{2n+2} = Bx_{2n+1} = x_{2n+2}$. Then

$$d(x_{2n+2}, x_{2n+1}) = d(Ax_{2n+3}, Bx_{2n+2})$$

$$\geq \phi(d(Ax_{2n+3}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+3}, x_{2n+2}))$$

$$= \phi(d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})).$$

By property $(C_2)$, $d(x_{2n+1}, x_{2n+2}) \geq h d(x_{2n+2}, x_{2n+3})$ or $x_{2n+2} = x_{2n+3}$. Thus $Ax_{2n+2} = Ax_{2n+3} = x_{2n+2}$ and $x_{2n+2}$ is a fixed point of $A$ also.

Suppose $x_n \neq x_{n+1}$ for all $n$. Then

$$d(x_{2n}, x_{2n+1}) = d(Ax_{2n+1}, Bx_{2n+2})$$

$$\geq \phi(d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}))$$

$$= \phi(d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})).$$
Again by (C2), \( d(x_{2n}, x_{2n+1}) \geq \frac{1}{h} d(x_{2n+1}, x_{2n+2}) \) or \( d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{h} d(x_{2n}, x_{2n+1}) \).

Also,

\[
\begin{align*}
\phi(d(Ax_{2n+2}, x_{2n+3}), d(Bx_{2n+2}, x_{2n+3}), d(x_{2n+3}, x_{2n+2})) \\
= \phi(d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+3}, x_{2n+2})).
\end{align*}
\]

By (C2) we get \( d(x_{2n+2}, x_{2n+3}) \leq \frac{1}{h} d(x_{2n+1}, x_{2n+2}) \). Induction gives

\[
\begin{align*}
d(x_{n+1}, x_{n+2}) & \leq \left( \frac{1}{h} \right)^{n+1} d(x_0, x_1). \\
\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2}) & \leq \sum_{n=1}^{\infty} \left( \frac{1}{h} \right)^{n+1} d(x_0, x_1) < \infty.
\end{align*}
\]

\( X \) is \( d \)-complete so \( x_n \to p \) as \( n \to \infty \) where \( p \in C \), since \( C \) is closed. We also have \( x_{2n} \to p \) and \( x_{2n+1} \to p \) as \( n \to \infty \). This gives \( Ax_{2n+1} \to p \) and \( Bx_{2n+2} \to p \) as \( n \to \infty \). Since \( p \in C \), \( p \in A(C) \) and \( p \in B(C) \), so there exist \( v, w \in C \) such that \( Av = p \) and \( Bw = p \). Now

\[
d(x_{2n}, p) = d(Ax_{2n+1}, Bw) \\
\geq \phi(d(Ax_{2n+2}, x_{2n+3}), d(Bw, w), d(x_{2n+3}, w)).
\]

Since \( \phi \) is lower-semicontinuous, letting \( n \to \infty \) gives \( d(p, p) \geq \phi(0, d(p, w), d(p, w)) \) and by (C2) we have \( 0 \geq h d(p, w) \). Hence \( p = w \). Also,

\[
d(p, x_{2n+1}) = d(Av, Bx_{2n+1}) \geq \phi(d(Av, v), d(Bx_{2n+2}, x_{2n+3}), d(v, x_{2n+1})).
\]

Letting \( n \to \infty \) gives \( d(p, p) \geq \phi(d(p, v), 0, d(v, p)) \) or, by (C2), \( 0 \geq h d(p, v) \). Hence \( p = v \).

Therefore, \( Ap = Av = p = Bw = Bp \).

**COROLLARY 2.** Let \( A \) and \( B \) map \( C \), a closed subset of \( X \), into (onto) \( X \) such that \( C \subseteq A(C) \), \( C \subseteq B(C) \), and \( d(Ax, By) \geq a d(Ax, x) + b d(By, y) + c d(x, y) \) for all \( x, y \in C \), where \( a, b, \) and \( c \) are non-negative real numbers with \( a < 1, b < 1, \) and \( a + b + c > 1 \). Then \( A \) and \( B \) have a common fixed point in \( C \).

The proof of Corollary 2 is identical to the proof of Corollary 2.3 in [9].

In [7], Kang defined \( \Phi^* \) to be the family of all real functions \( \varphi : (R^+)^3 \to R^+ \) satisfying condition (C1) and the following condition:

\[
(C_3) \quad \text{Let } v, w \in R^+ - \{0\} \text{ be such that either } v \geq \varphi(v, w, w) \text{ or } v \geq \varphi(w, v, w). \text{ Then } v \geq hw, \text{ where } \varphi(1, 1, 1) = h > 1. \text{ Kang showed that the family } \Phi^* \text{ is strictly larger than the family } \Phi.
\]

**THEOREM 7.** Let \((X, t, d)\) be a \( d \)-complete Hausdorff topological space where \( d \) is a continuous symmetric. If \( A \) and \( B \) are continuous mappings from \( C \), a closed subset of \( X \), into \( X \) such that \( C \subseteq A(C), C \subseteq B(C), \) and \( d(Ax, By) \geq a d(Ax, x) + b d(By, y) + c d(x, y) \) for all \( x, y \in C \), where \( a, b, \) and \( c \) are non-negative real numbers with \( a < 1, b < 1, \) and \( a + b + c > 1 \). Then \( A \) and \( B \) have a common fixed point.

**PROOF.** Let \( \{x_n\}_{n=0}^{\infty} \) be defined as in the proof of Theorem 6. If \( x_n = x_{n+1} \) for some \( n \) then \( A \) or \( B \) has a fixed point. Suppose \( x_n \neq x_{n+1} \) for all \( n \). As in the proof of Theorem 6, \( x_n \to p \) as \( n \to \infty \). Now \( \{x_{2n}\}_{n=0}^{\infty} \) and \( \{x_{2n+1}\}_{n=0}^{\infty} \) are subsequences of \( \{x_n\}_{n=0}^{\infty} \) and hence each converges to \( p \). Since \( A \) and \( B \) are continuous, \( Ax_{2n+1} = x_{2n} \to Ap \) and \( Bx_{2n+2} = x_{2n+1} \to Bp \). Limits in \( X \) are unique, because \( X \) is Hausdorff, so \( Ap = p = Bp \).
COROLLARY 3. Let $A$ and $B$ be continuous mappings from $C$, a closed subset of $X$, into $X$ satisfying $C \subseteq A(C)$, $C \subseteq B(C)$ and $d(Ax, By) \geq h \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all $x, y \in C$ with $x \neq y$ where $h > 1$. Then $A$ or $B$ has a fixed point or $A$ and $B$ have a common fixed point.

PROOF. Note that $\varphi(t_1, t_2, t_3) = h \min\{t_1, t_2, t_3\}, h > 1$ is in $\Phi^*$. Apply Theorem 7.

If $A = B$ in Corollary 3 we get a generalization of Theorem 3 in [9].

Boyd and Wong [10] call the collection of all real functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions $\Psi$:

(C4) $\psi$ is upper-semicontinuous and non-decreasing,

(C5) $\psi(t) < t$ for each $t > 0$.

THEOREM 8. Let $(X, t, d)$ be a $d$-complete symmetric Hausdorff topological space. If $A$ and $B$ are continuous mappings from $C$, a closed subset of $X$, into $X$ such that $C \subseteq A(C)$, $C \subseteq B(C)$, and $\psi(d(Ax, By)) \geq \min\{d(Ax, x), d(By, y), d(x, y)\}$ for all $x, y \in C$ where $\psi \in \Psi$ and $\sum_{n=0}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, then either $A$ or $B$ has a fixed point or $A$ and $B$ have a common fixed point.

PROOF. Let $\{x_n\}_{n=0}^{\infty}$ be defined as in the proof of Theorem 6. If $x_n = x_{n+1}$ for some $n$ then $A$ or $B$ has a fixed point. Suppose $x_n \neq x_{n+1}$ for all $n$. Then

$$\psi(d(x_{2n}, x_{2n+1})) = \psi(d(Ax_{2n+1}, Bx_{2n+2}))$$

$$\geq \min\{d(Ax_{2n+1}, x_{2n+1}), d(Bx_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\}$$

$$= \min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})\}$$

$$= d(x_{2n+1}, x_{2n+2})$$

since $\psi(t) < t$ for all $t > 0$.

Similarly, $d(x_{2n+2}, x_{2n+3}) \leq \psi(d(x_{2n+1}, x_{2n+2}))$ and hence $d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1}))$ for each $n$. Since $\psi$ is non-decreasing, $d(x_{n+1}, x_{n+1}) \leq \psi^n(d(x_0, x_1))$. Now

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=0}^{\infty} \psi^n(d(x_0, x_1)) < \infty.$$ 

The space $X$ is $d$-complete so there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. The mappings $A$ and $B$ are continuous so $Ax_{2n+1} = x_{2n} \to Ap$ and $Bx_{2n+2} = x_{2n+1} \to Bp$. Limits are unique so $Ap = p = Bp$.

REFERENCES


Special Issue on
Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

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- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
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- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
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<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>June 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>September 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>December 1, 2009</td>
</tr>
</tbody>
</table>

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