Internat. J. Math. & Math. Sci.
VOL. 18 NO. 3 (1995) 531-534

DIRECT SUMS OF J-RINGS AND RADICAL RINGS

XIUZHAN GUO

Department of Mathematics
China University of Mining and Technology
Xuzhou, Jiangsu 221008
China

(Received October 4, 1993 and in revised form May 20, 1994)

ABSTRACT. Let R be a ring, $J(R)$ the Jacobson radical of R, and P the set of potent elements of R. We prove that if R satisfies (*) given x,y in R there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that $x^m y = xy^n$, and if each $x \in R$ is the sum of a potent element and a nilpotent element, then N and P are ideals and $R = N \oplus P$. We also prove that if R satisfies (*), and if each $x \in R$ has a representation in the form $x = a + u$, where $a \in P$ and $u \in J(R)$, then P is an ideal and $R = J(R) \oplus P$.

KEY WORDS AND PHRASES. Periodic, potent, J-ring, radical ring, direct sum.
1991 AMS SUBJECT CLASSIFICATION CODE. 16U80.

1. INTRODUCTION.

Throughout this paper, for the ring R, $J(R)$ will denote the Jacobson radical of R, N the set of nilpotent elements of R, and P the set of potent elements of R — that is, the set of $x \in R$ for which there exists an integer $n = n(x) > 1$ such that $x^n = x$.

If $P = R$, we call R a J-ring; if $J(R) = R$, we call R a radical ring. A ring R is called periodic if for each $x \in R$ there exist distinct positive integers m,n for which $x^m = x^n$; following[2], R is called weakly periodic if each element is the sum of a potent element and a nilpotent element. It is known [1, Lemma 1] that all periodic rings are weakly periodic, but it is an open question whether weakly periodic rings must be periodic. In this paper, we consider the following condition:

(*) For each $x,y \in R$, there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that

$$x^m y = xy^n. \tag{1.1}$$

It is obvious that the above condition ($*$) is weaker than the condition $x^{n(x)} = x$ for all $x \in R$, since there exist non- J -rings satisfying ($*$). As an example, consider any zero ring R , i.e. $xy = 0$ for all $x, y \in R$.

2. MAIN RESULTS.

We begin with

LEMMA 1. Let R be a ring satisfying ($*$). Then P is a subring of R .

PROOF. If $a, b \in P$, then $a = a^m$, $b = b^n$ for some integers $m > 1$, $n > 1$. Let $e_a = a^{m-1}$ and $e_b = b^{n-1}$. Then

$$ae_a = a = e_a a \text{ and } e_a^2 = e_a,$$
$$be_b = b = e_b b \text{ and } e_b^2 = e_b.$$

Thus,

$$(e_a e_b - e_a e_b e_a)^2 = 0 = (e_b e_a - e_a e_b e_a)^2. \tag{2.1}$$

Let $x = e_a$ and $y = e_a e_b - e_a e_b e_a$ in (1.1). Using (2.1), we have

$$e_a e_b - e_a e_b e_a = e_a^{m_1}(e_a e_b - e_a e_b e_a) = e_a(e_a e_b - e_a e_b e_a)^{n_1} = 0$$

for some integers $m_1 > 1$ and $n_1 > 1$.

Similarly, we get $e_b e_a - e_a e_b e_a = 0$. Hence $e_a e_b = e_a e_b e_a = e_b e_a$. Let $e = e_a + e_b - e_a e_b$. Then

$$e^2 = e,\ ae = ea = a, \text{ and } be = eb = b. \tag{2.2}$$

Let $x = ab$ and $y = e$ in (1.1). Using (2.2), we have $ab = abe^{n_2} = (ab)^{m_2}e = (ab)^{m_2}$ for some integers $m_2 > 1$ and $n_2 > 1$. Similarly, we have $a - b = (a - b)^{m_3}$ for some integer $m_3 > 1$. Then $ab \in P$ and $a - b \in P$ as desired. The lemma is thus proved.

THEOREM 1. Let R be a weakly periodic ring satisfying($*$). Then N and P are ideals and $R = N \oplus P$.

PROOF. If $x, y \in R$ and $n = n(x,y) > 1$ and $m = m(x,y) > 1$ are such that $x^m y = xy^n$, then

$$x^{1+k(m-1)}y = xy^{1+k(n-1)} \qquad \text{for all positive integers } k. \tag{2.3}$$

It follows that

$$au = ua = 0 \text{ for all } a \in P \text{ and } u \in N. \tag{2.4}$$

This, together with Lemma 1 and the fact that $R = P + N$, shows that P is an ideal. To complete the proof, we need only show that N is an ideal, which by (2.4) amounts to showing that N is a subring.

Let u_1, $u_2 \in N$, and let $u_1 - u_2 = b + u$ for some $b \in P$ and $u \in N$. It follows from (2.4) that $(u_1 - u_2)^2 = (u_1 - u_2)u$, and hence that $(u_1 - u_2)^{k+1} = (u_1 - u_2)^k u$ for all $k \geq 1$. It is clear from (2.3) that $(u_1 - u_2)^k u = 0$ for some k , hence $u_1 - u_2 \in N$. A similar argument shows that $u_1 u_2 \in N$.

COROLLARY 1. Let R be a periodic ring satisfying($*$). Then N and P are both ideals and $R = N \oplus P$.

PROOF. Evident.

COROLLARY 2. Let R be a ring in which, given $x, y \in R$, there exist distinct integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that $x^m y = xy^n$. Then N and P are ideals and $R = N \oplus P$.

PROOF. For all $x \in R$, by hypothesis there exist distinct integers $m = m(x) > 1$ and $n = n(x) > 1$ such that $x^m x = xx^n$. Then R is periodic. Hence N and P are ideals of R and $R = N \oplus P$ by Corollary 1.

COROLLARY 3([1],[4],and[5]). Let R be a ring in which, given $x, y \in R$, there exists an integer $n = n(x,y) > 1$ such that $x^n y = xy^n$. Then N and P are ideals and $R = N \oplus P$.

PROOF. For all x, y in R, by hypothesis there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that

$$x^n y = xy^n \text{ and } (x^n)^m y = x^n y^m .$$

Then

$$x^{mn} y = x^n y^m = xy^{m+n-1} .$$

Since the equation $mn = m + n - 1$ has no integer solutions such that $m > 1$ and $n > 1$, there exist distinct integers $s = s(x,y) > 1$ and $t = t(x,y) > 1$ such that $x^s y = xy^t$. The corollary is thus proved by Corollary 2.

COROLLARY 4. Let R be a ring in which, given x, y in R, there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that $x^m y = xy = xy^n$. Then R is commutative.

PROOF. Obviously, R is periodic. Then N and P are ideals and $R = P \oplus N$ by Corollary 1. For all $x, y \in N$, there exists an integer $m = m(x,y) > 1$ such that

$$xy = x^m y = x^{m-1} xy = x^{2m-1} y = \cdots = 0.$$

Then N is a zero ring, and hence R is commutative.

REMARK. By the same process we used in proving the above results, we can prove

Let R be a ring in which, given x, y in R, there exist integers $m = m(x,y) > 1$ and $n = n(x,y) > 1$ such that $xy = x^m y^n$. Then (1) N and P are ideals with $N^2 = 0$; (2) $R = N \oplus P$ and R is commutative.

THEOREM 2. Let R be a ring satisfying (*). Suppose that each $x \in R$ has a representation in the form $x = a + u$, where $a \in P$ and $u \in J(R)$. Then P is an ideal and $R = J(R) \oplus P$.

PROOF. It is clear that $J(R) \cap P = \{0\}$. Since each $x \in R$ has a representation in the form $a + u$, where $a \in P$ and $u \in J(R)$, it suffices to prove that P is an ideal of R.

If $a \in P$ and $u \in J(R)$, then au, $ua \in J(R)$. Letting $x = e_a$ and $y = au$ in (1. 1), we have

$$au = e_a^m au = e_a (au)^n = (au)^n .$$

Since $au \in J(R)$ and $n > 1$, we have $au = 0$. Similarly, $ua = 0$. Then $PJ(R) = J(R)P = \{0\}$.

For all $a \in P$, $r \in R$, writing r in the form $r = r_1 + r_2$, where $r_1 \in P$, $r_2 \in J(R)$, we get $ra = (r_1 + r_2)a = r_1 a + r_2 a = r_1 a \in P$ and $ar = a(r_1 + r_2) = ar_1 + ar_2 = ar_1 \in P$. Then P is an ideal by Lemma 1. This completes the proof of Theorem 2.

We conclude with

THEOREM 3. Let R be a semisimple ring satisfying ($*$). Then R is isomorphic to a subdirect sum of fields.

PROOF. If R is a division ring, then, for all nonzero elements x, y in R, by (1. 1) we have $x^{m-1} = y^{n-1}$. Then $[x^{m-1}, y] = 0$ for all $x, y \in R$, so R is a field by a theorem of Herstein [3].

Suppose now that R is a primitive ring. Note that condition ($*$) is inherited by all subrings and all homomorphic images of R. Note also that no complete matrix ring $(D)_t$ over a division ring $D(t > 1)$ satisfies condition ($*$), as a consideration of $x = E_{12}$ and $y = E$ shows. Because of these facts and the structure theorem of primitive rings, we may assume that R is a division ring. Then R is a field.

If R is a semisimple ring, then R is isomorphic to a subdirect sum of primitive rings R_a each of which as a homomorphic image of R satisfies condition ($*$), so each R_a is a field. Thus, R is isomorphic to a subdirect sum of fields .

ACKNOWLEDGEMENT. The author wishes to express his indebtedness and gratitude to the referee for his helpful suggestions and valuable comments.

REFERENCES

1. BELL, H. E. A commutativity study for periodic rings, Pacific J. Math. 70 (1977), 29—36.
2. GROSEN, J. , TOMINAGA, H. , and YAQUB, A. On weakly periodic rings, periodic rings, and commutativity theorems, Math. J. Okayama Univ. 32 (1990), 77—81.
3. HERSTEIN, I. N. Two remarks on the commutativity of rings, Canad. J. Math. 7(1955), 411-412.
4. LUH, J. On the structure of pre-J-rings, Hung-ching Chow Sixty-fifth Anniversary Volume, Math. Res. Center Nat. Taiwan Univ. , Taipei, 1967.
5. YAQUB, A. The structure of pre-p^k-rings and generalized pre-p-rings, Amer. Math. Monthly 71 (1964), 1010—1014.