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OPERATORS ACTING ON CERTAIN BANACH SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $\mathcal{X}$ be a reflexive Banach space of functions analytic on a plane domain Ω such that for every λ in Ω the functional of evaluation at λ is bounded . Assume further that $\mathcal{X}$ contains the constants and M_z , multiplication by the independent variable z, is a bounded operator on $\mathcal{X}$. We give sufficient conditions for M_z to be reflexive. In particular, we prove that the operators M_z on $E^p(\Omega)$ and certain $H_\alpha^p(\beta)$ are reflexive. We also prove that the algebra of multiplication operators on Bergman spaces is reflexive, giving a simpler proof of a result of Eschmeier.

KEY WORDS AND PHRASES. Banach spaces of analytic functions, Smirnov domain, bounded point evaluation. 1991 SUBJECT CLASSIFICATION. Primary 47B37; Secondary 47A25.

1 INTRODUCTION.

Let Ω be a bounded domain in the complex plane **C**. Suppose $\mathcal{X}$ is a reflexive Banach space consisting of functions that are analytic on Ω such that $1 \in \mathcal{X}$, for each λ in Ω, the functional e(λ): $\mathcal{X} \longrightarrow \mathbf{C}$ of evaluation at λ given by $e(\lambda)(f) = < f , e(\lambda) > = f(\lambda)$ is bounded, and if f $\in \mathcal{X}$ then zf $\in \mathcal{X}$. Note that the last condition allows us to define $M_z : \mathcal{X} \longrightarrow \mathcal{X}$ by $M_z f = zf$, f $\in \mathcal{X}$. It is easy to see that M_z is actually a bounded operator on $\mathcal{X}$. If $\mathcal{X}$ is a Hilbert space, the operator M_z and many of its properties have been studied in Shields and Wallen [1]; Bourdon and Shapiro [2]. We would like to give some sufficient conditions so that the operator M_z becomes reflexive.

Let Ω be a bounded open set in **C** and let p be a real number with $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the L^p-space of the 2-dimensional Lebesgue measure restricted to Ω. The space of analytic functions on Ω is denoted by $H(\Omega)$ and as usual $H^\infty(\Omega)$ is the Banach space of all bounded functions analytic on Ω equipped with the supremum norm. Each function $f \in H^\infty(\Omega)$ induces a bounded operator $M_f : L_a^p(\Omega) \longrightarrow L_a^p(\Omega),\ g \longrightarrow fg$, where $L_a^p(\Omega)$ is the subspace of $L^p(\Omega)$ consisting of all analytic functions. This space is called the *Bergman space*.

In this article we shall prove that the algebra $B = \{M_f \mid f \in H^\infty(\Omega)\}$ is reflexive. We give a shorter proof of a result of J. Eschmeier [3] in case Ω is a plane domain.

2 PRELIMINARIES.

In this section we make a few definitions and set our notation straight. If G is a bounded domain in the plane, the Carathéodory hull ($\mathcal{C}$-hull) of G is the complement of the closure of the unbounded component of the complement of the closure of G. It can be described as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leq sup\{|p(z)| : z \in G\}$ for all polynomials p. An open set G is called a *Carathéodory domain* if it is equal to the component of the Carathéodory hull of G that contains it.

For the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators on a Banach space $\mathcal{X}$, the weak operator topology (WOT) is the one in which a net A_α converges to A if $A_\alpha x \longrightarrow Ax$ weakly, $x \in \mathcal{X}$.

A complex valued function ϕ on Ω for which $\phi f \in \mathcal{X}$ for every $f \in \mathcal{X}$ is called a *multiplier* of $\mathcal{X}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{X})$. Because M_z is a bounded operator on $\mathcal{X}$, the adjoint $M_z^*: \mathcal{X}^* \longrightarrow \mathcal{X}^*$ satisfies $M_z^* e(\lambda) = \lambda e(\lambda)$. In general each multiplier ϕ of $\mathcal{X}$ determines a multiplication operator M_ϕ defined by $M_\phi f = \phi f$, $f \in \mathcal{X}$. Also $M_\phi^* e(\lambda) = \phi(\lambda) e(\lambda)$. It is well known that each multiplier is a bounded analytic function, Shields and Wallen [1]. Indeed $|\phi(\lambda)| \leq ||M_\phi||$ for each λ in Ω. Also $M_\phi 1 = \phi \in \mathcal{X} \subset H(\Omega)$. So ϕ is a bounded analytic function.

Recall that if $\mathcal{E}$ is a subalgebra of $\mathcal{B}(\mathcal{X})$ containing the identity operator, then Lat$(\mathcal{E})$ is by definition the lattice of all invariant subspaces of $\mathcal{E}$, and Alg Lat$(\mathcal{E})$ is the algebra of all operators B in $\mathcal{B}(\mathcal{X})$ such that Lat$(\mathcal{E}) \subset Lat(B)$. We say that $\mathcal{E}$ is *reflexive* if $\mathcal{E} =$ Alg Lat$(\mathcal{E})$. Obviously a reflexive algebra $\mathcal{E}$ is (WOT)-closed. An operator A in $\mathcal{B}(\mathcal{X})$ is said to be *reflexive* if Alg Lat(A)= W(A), where W(A) is the smallest subalgebra of $\mathcal{B}(\mathcal{X})$ that contains A and the identity I and is closed in the weak operator topology.

Let $A \in Alg\ Lat(M_z)$ and let $\mathcal{M}$ be a weak star closed invariant subspace of M_z^* in $\mathcal{X}^*$. Then ${}^\perp\mathcal{M} \in Lat(M_z)$ and hence ${}^\perp\mathcal{M} \in Lat(A)$. Therefore, $({}^\perp\mathcal{M})^\perp \in Lat(A^*)$. Since $\mathcal{M}$ is weak star closed, $\mathcal{M} \in Lat(A^*)$. Now the one-dimensional span of $e(\lambda)$ is invariant under M_z^*. Therefore, it is invariant under A^*. We write $A^* e(\lambda) = \phi(\lambda) e(\lambda)$, $\lambda \in \Omega$. So $< f\ ,\ A^* e(\lambda) > = \phi(\lambda) f(\lambda)$; $\lambda \in \Omega$. Using the Hahn-Banach theorem we see that the linear span of $\{e(\lambda)\}_{\lambda \in \Omega}$ is weak star dense in $\mathcal{X}^*$. Thus $\phi \in \mathcal{M}(\mathcal{X})$ and $A = M_\phi$.

3 REFLEXIVITY.

In this section we consider a Banach space of functions analytic on a Carathéodory domain and give sufficient conditions for the operator of multiplication to be reflexive. A circular domain is also considered.

THEOREM 1. Let Ω be a Carathéodory domain each point of which is a bounded point evaluation for a reflexive Banach space $\mathcal{X}$ of functions analytic on Ω which contains the constant functions and admits M_z as a bounded operator. Furthermore, if $||M_p|| \leq C||p||_\Omega$ for every polynomial p, then M_z is reflexive.

PROOF. Let A $\in$ Alg Lat(M_z). Then $A = M_\phi$ for some multiplier $\phi \in H^\infty(\Omega)$. Let $\{p_n\}$ be a sequence of polynomials such that $\sup||p_n||_\Omega \leq M$ for some constant M and $p_n(z) \longrightarrow \phi(z)$, $z \in \Omega$. Then $||M_{p_n}|| \leq C||p_n||_\Omega \leq CM$. Since $\mathcal{X}$ is reflexive, the unit ball of $\mathcal{X}$ is weakly compact. Therefore, the unit ball of $\mathcal{B}(\mathcal{X})$ is (WOT) compact. We may assume , by passing to a subsequence if necessary, that $M_{p_n} \longrightarrow X$ (WOT) for some operator X. Thus $M_{p_n}^* e(\lambda) \longrightarrow X^* e(\lambda)$ in the weak star topology. On the other hand $M_{p_n}^* e(\lambda) = p_n(\lambda) e(\lambda) \longrightarrow \phi(\lambda) e(\lambda) = M_\phi^* e(\lambda)$ in the weak star topology for every $\lambda \in \Omega$. Therefore, $X^* e_\lambda = M_\phi^* e_\lambda$ and thus $X^* = M_\phi^*$. Hence $X = M_\phi$ on $\mathcal{X}$, which implies that $A \in W(M_z)$ and M_z is reflexive.□

Now we use the technique of the proof of Theorem 1 to give a short proof of a result of Eschmeier [3]. We let $B = \{M_f | f \in H^\infty(\Omega)\}$, where Ω is a bounded domain and M_f acts on $L_a^p(\Omega)$.

THEOREM 2. The algebra B is reflexive.

PROOF. Clearly B$\subseteq$ Alg Lat(B). Let A $\in$ Alg Lat(B). Because the one dimensional span of e(λ) is invariant under M_f^* for all f in $H^\infty(\Omega)$, it is invariant under A^*, and therefore $A = M_\phi$ for some multiplier ϕ. Thus B is a reflexive algebra.□

Next we give a few examples of Banach spaces satisfying the hypothesis of Theorem 1.

EXAMPLE 3. Let Ω be an arbitrary simply connected Smirnov domain. Let $1 < p < \infty$. Define $E^p(\Omega)$ to be the set of all analytic functions f on Ω such that there exists a sequence of rectifiable Jordan curves $C_1, C_2, \cdots$ in Ω, tending to the boundary in the sense that C_n eventually surrounds each compact subdomain of Ω , such that $\int_{C_n} |f(z)|^p |dz| \leq M < \infty$. For a good source on $E^p(\Omega)$ see Duren [4, Chapter 10]. Every function f of class $E^p(\Omega)$ has a nontangential limit almost everywhere on $\partial\Omega$, which does not vanish on a set of positive measure unless $f(z) \equiv 0$. Furthermore, $\int_{\partial\Omega} |f(z)|^p |dz| < \infty$. It is convenient to identify $E^p(\Omega)$ with its set of boundary functions. Thus $E^p(\Omega)$ is a closed subspace of $L^p(\partial\Omega)$ which contains the set of all polynomials, and hence its closure. Hence $E^p(\Omega)$ is a reflexive Banach space.

Clearly M_z is bounded and $||M_p|| \leq ||p||_\Omega$ for all polynomials p. Now we show that each point of Ω is a bounded point evaluation for $E^p(\Omega)$ For a fixed z in Ω, choose $C > 0$ such that dist(z,$\partial\Omega$) $\geq$ C Let f $\in E^p(\Omega)$ Then $f^p \in E^1(\Omega)$ and it has a Cauchy representation

$$f^p(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f^p(\varsigma)}{\varsigma - z} d\varsigma\ , \quad z \in \Omega$$

Therefore $|f(z)|^p \leq (1/2\pi C)||f||^p$ Thus each point of Ω is a bounded point evaluation for $E^p(\Omega)$. Finally, by Theorem 3 1, M_z is reflexive

Further examples of Banach spaces satisfying the hypothesis of Theorem 1 will be presented We also deduce that M_z acting on these spaces are reflexive. We begin with a definition.

DEFINITION 4. Let $1 < p < \infty$ and let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. We consider the space of sequences $f = \{\hat{f}(n)\}$ such that

$$||f||_p^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p[\beta(n)]^p < \infty.$$

We shall use the formal notation $f(z) = \sum_{n=o}^{\infty} \hat{f}(n)z^n$ for z $\in$ **D** the unit disc in **C** (See Shields [5] for p=2.). Let $H^p(\beta) = \{f | f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n;\ ||f||_p < \infty\}$ and $H_a^p(\beta) = \{f \in H^p(\beta) |\ f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \textit{ is convergent in } \mathbf{D}\}$.

REMARK 5. Define the σ-finite measure μ on the positive integers by $\mu(K) = \sum_{n\in K} \beta(n)^p$, $K \subseteq$ **N**. Because $H^p(\beta) \cong L^p(\mu)$ we conclude that $H^p(\beta)$ is indeed a reflexive Banach space.

REMARK 6. If $\{\beta(n+1)/\beta(n)\}$ is bounded, the operator of multiplication by z is a bounded operator on $H^p(\beta)$. Indeed $||M_z|| = sup_n \frac{\beta(n+1)}{\beta(n)}$.

In the following examples let q be the conjugate of p (1/p + 1/q =1).

EXAMPLE 7. Let $\{1/\beta(n)\} \in \ell^q$. If f $\in H^p(\beta)$ and $\lambda \in$ **D**, we have

$$|f(\lambda)| = |\sum_{n=0}^{\infty} \hat{f}(n)\lambda^n| \leq (\sum_{n=0}^{\infty} |\hat{f}(n)|^p[\beta(n)]^p)^{1/p}(\sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q})^{1/q}. \tag{1}$$

Therefore, f is analytic and $||f||_{\mathbf{D}} \leq ||\{\frac{1}{\beta(n)}\}||_q\ ||f||_p$. We conclude that $H_a^p(\beta) = H^p(\beta) \subset H^\infty$. Furthermore, each point of **D** is a bounded point evaluation for $H^p(\beta)$ and also convergence in $H^p(\beta)$ implies uniform convergence on **D**.

EXAMPLE 8. In Example 7 assume $\beta(n) \geq 1$ for all n. In this case, it follows from (1) that $||f||_K \leq C||f||_p$ for any compact K $\subset$ **D**, where C depends on K.

EXAMPLE 9. Let $p > 1$. Also suppose that $sup_n \frac{\beta(n+1)}{\beta(n)} = 1$ (e.g. $\beta(n) = 1$ or β (n) =1+1/n). It can easily be seen that $\overline{\mathbf{D}} = \sigma(M_z)$. Since M_z is a contraction, $\overline{\mathbf{D}}$ is a spectral set for M_z and $||M_p|| \leq ||p||_{\mathbf{D}}$ for every polynomial p. By Theorem 1, M_z acting on $H_a^p(\beta)$ is reflexive.

The domains considered in Theorem 1 were Carathéodory domains. We now extend the conclusion of this Theorem to a circular domain, that is, any domain obtained by removing a finite number of disjoint subdiscs from the open unit disc. In Seddighi and Yousefi [6] we have proved the analogue of the following theorem for a Hilbert subspace of H(Ω). For the proof combine the techniques of the proof of Theorem 1 with Seddighi and Yousefi [6, Theorem 5.1].

THEOREM 10. Let Ω be a circular domain each point of which is a bounded point evaluation for a reflexive Banach subspace $\mathcal{X}$ of H(Ω) which contains the constants and admits multiplication by the independent variable z, M_z, as a bounded operator. Furthermore, suppose that $||M_p|| \leq ||p||_\Omega$ for every polynomial p. Then M_z is reflexive.

We present an example of a Banach space satisfying the hypothesis of Theorem 10.

EXAMPLE 11. Let Ω be a circular domain and $1 < p < \infty$. Since $L_a^p(\Omega)$ is closed in $L^p(\Omega)$, $L_a^p(\Omega)$ is reflexive. By Lemma 3.7 of Garnett [7] every point of Ω is a bounded point evaluation for $L_a^p(\Omega)$. It is also clear that $||M_p|| \leq ||p||_\Omega$ for every polynomial p. By Theorem 4 the multiplication operator M_z on $L_a^p(\Omega)$ is reflexive.

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