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LOCALLY COMPACT SEMI-ALGEBRAS GENERATED BY A COMMUTING OPERATOR FAMILY

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ABSTRACT. Conditions are provided for the local compactness of the closed semi-algebra generated by a finite collection of commuting bounded linear operators with equibounded iterates in terms of their joint spectral properties.

Key Words and Phrases: Semi-algebra, Locally Compact, Radon-Nikoloskii Operator.

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1. INTRODUCTION.

The notion of an abstract locally compact semi-algebra was introduced by Bonsall [1], who developed a theory of its algebraic properties. Bonsall and Tomiuk [2] showed that a compact linear operator with spectral radius 1 and no generalized eigenvectors corresponding to eigenvalues on the unit circle generates a locally compact closed semi-algebra. This result was generalized by Kaashoek and West [5] who obtained a complete characterization of locally compact closed semi-algebras generated by a single operator t with equibounded iterates in terms of its spectrum. Basically, an operator t with equibounded iterates generates a locally compact closed semi-algebra if and only if its spectrum on the unit circle consists of a finite number of eigenvalues with finite dimensional eigenspaces. In this article, arising from a Ph.D. thesis [6], we generalize these results to closed semi-algebras generated by a finite number of commuting operators with equibounded iterates. Banach algebra techniques are used in the proofs. Our results are then applied to the closed semi-algebra generated by a finite set of commuting scalar-type spectral operators [3]. Some illustrative examples are provided.

In Section 2 we prove the local compactness of the closed semi-algebra generated by a finite number of commuting Radon-Nikolskii operators with equibounded iterates and a common eigenvector at the eigenvalue 1. In Section 3 we give various equivalent characterizations for the local compactness of the closed semi-algebra generated by a finite set of commuting operators with equibounded iterates. Section 4 is devoted to examples.

Throughout this article, if X is a complex Banach space, we denote the Banach algebra of bounded linear operators on X by $\mathcal{L}(X)$. We denote the spectrum of an operator t by $\sigma(t)$, its spectral radius by $r(t)$ and the identity operator by I.

2. SEMI-ALGEBRAS GENERATED BY RADON-NIKOLSKII OPERATORS.

Bonsall and Tomiuk [2] have proved that a closed semi-algebra generated by an appropriately normalized compact operator, which is not quasinilpotent, is locally compact. In this section we prove that the conclusion still holds for a closed semi-algebra generated by a finite number of commuting Radon-Nikolskii operators, provided they are normalized and have a common eigenvector with eigenvalue 1.

Let us first give the necessary definitions.

By a **semi-algebra** we denote a subset $\mathcal{A}$ of a Banach algebra Z (in the present context, $Z = \mathcal{L}(X)$) which is closed with respect to multiplication and such that $\alpha a + \beta b \in \mathcal{A}$ for all $\alpha, \beta \geq 0$ and $a, b \in \mathcal{A}$. We call the semi-algebra $\mathcal{A}$ **locally compact** if $\mathcal{A} \neq \{0\}$ and the set $\{a \in \mathcal{A} : \|a\| \leq 1\}$ is compact in the norm topology of the Banach algebra Z.

Let $t_1, \cdots, t_m$ be commuting operators in $\mathcal{L}(X)$. Let us denote the semi-algebra of operators of the type

$$P(t_1, \cdots, t_m) = \sum_{i_1+\cdots+i_m=1}^{n} \alpha_{i_1,i_2,\cdots,i_m} t_1^{i_1} t_2^{i_2} \cdots t_m^{i_m},$$

where $\alpha_{i_1,\cdots,i_m} \in \mathbf{R}^+$, by $\mathcal{P}(t_1, \cdots, t_m)$, and its closure in the uniform operator topology by $\mathcal{A}(t_1, \cdots, t_m)$.

An operator $t \in \mathcal{L}(X)$ is called a **Radon-Nikolskii operator** if it has the form $t = s + k$ where $s, k \in \mathcal{L}(X)$, $r(s) < r(t)$ and k is compact. An equivalent way to define a Radon-Nikolskii operator is to write it in the form $t = tp + t(I - p)$ where p is a bounded projection of finite rank commuting with t, t restricted to the range of p has its spectrum on the circle $\{\lambda \in \mathbf{C} : |\lambda| = r(t)\}$, and $r(t(I - p)) < r(t)$.

THEOREM 2.1. *Let s and t in $\mathcal{L}(X)$ be commuting Radon-Nikolskii operators with $r(s) = r(t) = 1$. Assume there exists a non-zero $x \in X$ such that $sx = tx = x$. Then the closed semi-algebra $\mathcal{A}(s, t)$ is locally compact.*

PROOF. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{A}(s, t)$ of unit norm. Since $\mathcal{A}(s, t)$ is the closure of $\mathcal{P}(s, t)$, there exists a sequence $\{b_n\}_{n=1}^{\infty}$ of operators in $\mathcal{P}(s, t)$ of unit norm such that $\|a_n - b_n\| < \frac{1}{n}$ for all $n \in \mathbf{N}$. Writing

$$b_n = \sum_{i+j=1}^{\infty} \alpha_{n,i,j} s^i t^j,$$

where $\alpha_{n,i,j} \geq 0$ and $\alpha_{n,i,j} = 0$ for sufficiently large $i + j$, we have for each $n \in \mathbf{N}$

$$\sum_{i+j=1}^{n} \alpha_{n,i,j} \leq \|b_n\| \leq 1.$$

Therefore, by the diagonal process, there exists a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers such that

$$\lim_{k\to\infty} \alpha_{n_k,i,j} = \alpha_{i,j} \in \mathbf{R}^+.$$

Hence, for every positive integer N we have

$$\sum_{i+j=1}^{N} \alpha_{i,j} = \lim_{k\to\infty} \sum_{i+j=1}^{N} \alpha_{n_k,i,j} \leq 1,$$

and therefore

$$\sum_{i+j=1}^{\infty} \alpha_{i,j} \leq 1.$$

Since s and t are Radon-Nikolskii operators, there exist bounded linear projections p and q of finite rank such that $sp = ps$, $tq = qt$, $r(s(I-p)) < 1$ and $r(t(I-q)) < 1$. More precisely, $p = \frac{1}{2\pi i}\int_\Gamma (zI-s)^{-1}dz$ and $q = \frac{1}{2\pi i}\int_\Gamma (zI-t)^{-1}dz$, where Γ is the positively oriented circle with radius ϵ about 1 with ϵ small enough. As a result, p, q, s and t commute with each other.

Now consider

$$b = \sum_{i+j=1}^{\infty} \alpha_{i,j} s^i (I-p) t^j (I-q).$$

Then, because $r(s(I-p)) < 1$ and $r(t(I-q)) < 1$, the series defining b is absolutely convergent in the norm of $\mathcal{L}(X)$. Similarly, the series in

$$b_n(I-p)(I-q) = \sum_{i+j=1}^{\infty} \alpha_{n,i,j} s^i (I-p) t^j (I-q)$$

is absolutely convergent in the norm of $\mathcal{L}(X)$ and we have the estimate

$$\sum_{i+j=1}^{\infty} |\alpha_{n,i,j} - \alpha_{i,j}| \|s^i(I-p)t^j(I-q)\| \le 2\left(\sum_{i=0}^{\infty} \|s^i(I-p)\|\right)\left(\sum_{j=0}^{\infty} \|t^j(I-q)\|\right) < \infty.$$

Since this upper bound is independent of n, we may apply the principle of dominated convergence and derive that

$$\lim_{k\to\infty} \|b_{n_k}(I-p)(I-q) - b\| = 0.$$

Consider the finite-dimensional subspace $Y = p[X] + q[X]$ of X. Since p and q commute with $s^i t^j$, Y is an invariant subspace of b_n. Let c_n be the restriction of b_n to Y. Then $\|c_n\| \le \|b_n\| \le 1$, $n \in \mathbf{N}$. Since the unit ball in $\mathcal{L}(Y)$ is compact, there exist a strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, a subsequence of the sequence above, and some $c \in \mathcal{L}(Y)$ such that $\|c_{n_k} - c\| \to 0$ as $k \to \infty$. But then using $c_{n_k} = b_{n_k}(p+q-pq)$ we get

$$\lim_{k\to\infty} \|b_{n_k}(p+q-pq) - c(p+q-pq)\| = 0.$$

Since

$$a_{n_k} - [b + c(p+q-pq)] = \{a_{n_k} - b_{n_k}\} + \{b_{n_k}(I-p)(I-q) - b\} + b_{n_k}(p+q-pq),$$

we find

$$\lim_{k\to\infty} \|a_{n_k} - [b + c(p+q-pq)]\| = 0,$$

whence $b + c(p+q-pq) \in \mathcal{A}(s,t)$. Thus every bounded sequence in $\mathcal{A}(s,t)$ has a convergent subsequence and therefore $\mathcal{A}(s,t)$ is locally compact. ∎

REMARK. If t is a Radon-Nikolskii operator and $1 = r(t) \in \sigma(t)$, there exists a non-zero $x \in X$ such that $tx = x$. Then, applying Theorem 2.1 for $s = t$, we find that $\mathcal{A}(t)$ is locally compact.

REMARK. The conclusion of Theorem 2.1 still holds for a semi-algebra generated by a finite number of commuting Radon-Nikolskii operators under the same conditions. We state this fact in the following theorem. We have omitted its proof, because it is very similar to the proof of Theorem 2.1.

THEOREM 2.2. *Let $t_1, \cdots, t_m$ be commuting Radon-Nikolskii operators in $\mathcal{L}(X)$, and let the spectral radii $r(t_i) = 1$ for $i = 1, 2, \cdots, m$. Suppose there exists a non-zero $x \in X$ such that $t_1 x = t_2 x = \cdots = t_m x = x$. Then the closed semi-algebra $\mathcal{A}(t_1, t_2, \cdots, t_m)$ generated by $t_1, t_2, \cdots, t_m$ is locally compact.*

If Σ is a commuting semigroup of operators in $\mathcal{L}(X)$ and K is a compact convex subset of X not containing 0 such that $\{tx : x \in K, t \in \Sigma\} \subset K$, then the Kakutani-Markov theorem ([**4**], Theorem V 10.6) guarantees the existence of some $x \in K$ such that $tx = x$ for every $t \in \Sigma$. Hence, we can restate the above theorem in the following form.

THEOREM 2.3. *Let Σ be a semigroup of commuting Radon-Nikolskii operators in $\mathcal{L}(X)$ with a finite number of generators $t_1, t_2, \cdots, t_m$. Let K be a compact convex subset of X not containing 0, and let Σ map K into itself. If the spectral radii $r(t_i) = 1$ for $i = 1, 2, \cdots, m$, then the smallest closed semi-algebra $\mathcal{A}(\Sigma)$ containing Σ is locally compact.*

3. PRIME LOCALLY COMPACT SEMI-ALGEBRAS.

Kaashoek and West [**5**] have given necessary and sufficient conditions for an operator to generate a locally compact semi-algebra. Assuming the operator to have equibounded iterates, they have given the conditions in terms of the spectrum of the operator. In this section we generalize some of their results to semi-algebras generated by finitely many commuting operators.

DEFINITION. Let $t_1, \cdots, t_m$ be a finite number of commuting operators in $\mathcal{L}(X)$. We say that the m-tuple $(t_1, \cdots, t_m)$ has **equibounded iterates** if the set

$$\Lambda(t_1, \cdots, t_m) = \{t_1^{i_1} \cdots t_m^{i_m}\}_{i_1 + \cdots + i_m = 1}^{\infty}$$

is uniformly bounded in $\mathcal{L}(X)$. This is equivalent to saying that each one of the operators t_i has equibounded iterates.

DEFINITION. Let A be a semi-algebra. Then A is called **strict** if $A \cap (-A) = \emptyset$. The semi-algebra A is called **semisimple** if $a \neq 0$ implies $a^2 \neq 0$ for each $a \in A$. The semi-algebra $\mathcal{A}$ is called **prime** if for all non-zero $a, b \in A$ the element $ab \neq 0$.

Let $\mathcal{A}(s,t)$ be the closed semi-algebra generated by the operators s and t. Let

$$\Gamma = \{p(s,t) \in \mathcal{P}(s,t) : 0 < p(1,1) \leq 1\}.$$

Since the convex hull $co\{\Lambda(s,t)\}$ of $\Lambda(s,t)$ is the set

$$\{p(s,t) \in \mathcal{P}(s,t) : p(1,1) = 1\},$$

we have

$$\Lambda(s,t) \subset co\{\Lambda(s,t)\} \subset \Gamma.$$

It follows trivially that the conditional compactness of any of these three sets implies the conditional compactness of the others. We state this fact in the following lemma (cf. [5], Lemma 6).

LEMMA 3.1. *Let s and t be commuting operators in $\mathcal{L}(X)$. Then the following statements are equivalent:*

(1) $\Lambda(s,t)$ is conditionally compact;

(2) $co\{\Lambda(s,t)\}$ is conditionally compact;

(3) Γ is conditionally compact.

Let $\mathcal{B}(s,t)$ denote the smallest Banach subalgebra of $\mathcal{L}(X)$ containing I, s and t, and let $\mathcal{M}$ denote the set of multiplicative linear functionals in $\mathcal{B}(s,t)$. Then we have the following lemma.

LEMMA 3.2. *Let s and t be commuting operators in $\mathcal{L}(X)$. Let $1 \in \sigma(s) \cap \sigma(t)$, and let $s\hat{}$ and $t\hat{}$ denote the Gel'fand transforms of s and t, respectively. Assume that there exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$. Then*

(1) if the spectral radii $r(s) \leq 1$ and $r(t) \leq 1$, then $r(p(s,t)) \leq p(1,1)$ for all $p \in \mathcal{P}(s,t)$;

(2) if $\Lambda(s,t)$ is uniformly bounded, then there is a positive constant K, independent of $p(s,t)$, such that

$$\|p(s,t)\| \leq Kr(p(s,t)).$$

PROOF. (1) Let $p(s,t) = \sum_{i+j=1}^{k} \alpha_{i,j} s^i t^j$, $\alpha_{i,j} \geq 0$. Then

$$\begin{aligned} r(p(s,t)) &= \sup\{|\lambda| : \lambda \in \sigma(p(s,t))\} \\ &\leq \sup\{|\lambda| : \lambda \in p(\sigma(s) \times \sigma(t))\} \\ &= \sup\{|p(\alpha,\beta)| : (\alpha,\beta) \in \sigma(s) \times \sigma(t)\} \\ &\leq \sup\{\sum_{i+j=1}^{k} \alpha_{i,j} |\alpha^i||\beta^j| : \alpha \in \sigma(s), \beta \in \sigma(t)\} \\ &\leq \sum_{i+j=1}^{k} \alpha_{i,j} = p(1,1). \end{aligned}$$

Since there exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$, we also have

$$\sum_{i+j=1}^{k} \alpha_{i,j} = p(s,t)\hat{}(M) \leq r(p(s,t)) \leq p(1,1).$$

(2) Since $\Lambda(s,t)$ is uniformly bounded, we have $r(s) \leq 1$ and $r(t) \leq 1$. Also,

$$\begin{aligned} \|p(s,t)\| = \|\sum_{i+j=1}^{k} \alpha_{i,j} s^i t^j\| &\leq \sum_{i+j=1}^{k} \alpha_{i,j} \|s^i t^j\| \\ &= K \sum_{i+j=1}^{k} \alpha_{i,j} = K\, p(1,1), \end{aligned}$$

and using (1) we get

$$\|p(s,t)\| \leq Kr(p(s,t)),$$

which completes the proof. ∎

COROLLARY 3.3. *Let s and t be commuting operators in $\mathcal{L}(X)$, and let $\Lambda(s,t)$ be uniformly bounded. If there exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$, then there exists a constant $K > 0$ such that*

$$\|u\| \leq Kr(u), \ u \in \mathcal{A}(s,t).$$

PROOF. Since $\mathcal{A}(s,t)$ is a commutative semi-algebra, the spectral radius is a continuous function on $\mathcal{A}(s,t)$ in the uniform operator topology. Hence the result follows from Lemma 3.2. ∎

LEMMA 3.4. *Let s and t be commuting operators in $\mathcal{L}(X)$, and let $s\hat{}(M) = t\hat{}(M) = 1$ for some $M \in \mathcal{M}$. Let $\|a\| \leq K$ for every $a \in \Lambda(s,t)$. Then for each $a \in \mathcal{A}(s,t)$ there exists $\alpha \in \mathbf{R}^+$ such that $a\hat{}(M) = \alpha$ and $\alpha \geq K^{-1}\|a\|$.*

PROOF. Given $a \in \mathcal{A}(s,t)$ and $\epsilon > 0$, there exists $b \in \mathcal{P}(s,t)$ such that

$$\|b - a\| \leq \epsilon\|a\|.$$

Obviously, we have $a\hat{}(M) = \alpha$ for some $\alpha \in \mathbf{R}^+$. Let

$$b = \sum_{i+j=1}^{k} \alpha_{i,j} s^i t^j.$$

Then $(b-a)^\wedge(M) = \sum_{i+j=1}^{k} \alpha_{i,j} - \alpha$, and

$$\left| \sum_{i+j=1}^{k} \alpha_{i,j} - \alpha \right| \leq \|b-a\| \leq \|a\|.$$

Therefore, $\alpha \geq \sum_{i+j=1}^{k} \alpha_{i,j} - \epsilon\|a\|$. Also,

$$(1-\epsilon)\|a\| \leq \|b\| \leq \sum_{i+j=1}^{k} \alpha_{i,j}\|s^i t^j\| \leq K \sum_{i+j=1}^{k} \alpha_{i,j},$$

so that

$$\alpha \geq K^{-1}(1-\epsilon)\|a\| - \epsilon\|a\|.$$

Since ϵ is arbitrary, the result follows.

LEMMA 3.5. *Let s and t be commuting operators in $\mathcal{L}(X)$, let $r(s) = r(t) = 1$, and let there exist an $x \in X$ such that $sx = tx = x$. Then there exists an element $M \in \mathcal{M}$ such that $s^\wedge(M) = t^\wedge(M) = 1$.*

PROOF. Since $(s+t)x = sx + tx = 2x$, $2 \in \sigma(s+t)$ and hence there exists an element $M \in \mathcal{M}$ such that $(s+t)^\wedge(M) = 2$. But $|s^\wedge(M)| \leq 1$ and $|t^\wedge(M)| \leq 1$, since $r(s) = r(t) = 1$. Hence,

$$2 = (s+t)^\wedge(M) = s^\wedge(M) + t^\wedge(M) \leq |s^\wedge(M)| + |t^\wedge(M)| \leq 2,$$

whence $s^\wedge(M) = t^\wedge(M) = 1$. ∎

We now prove the main theorem of this section.

THEOREM 3.6. *Let s and t be commuting operators in $\mathcal{L}(X)$ such that $r(s) = r(t) = 1$ and $1 \in \sigma(s) \cap \sigma(t)$. Then the following statements are equivalent:*

(1) The set $\Lambda(s,t)$ is conditionally compact and there exists an element $M \in \mathcal{M}$ such that $s^\wedge(M) = t^\wedge(M) = 1$.

(2) The semi-algebra $\mathcal{A}(s,t)$ is locally compact, $\Lambda(s,t)$ is uniformly bounded and there exists an element $M \in \mathcal{M}$ such that $s^\wedge(M) = t^\wedge(M) = 1$.

(3) The semi-algebra $\mathcal{A}(s,t)$ is locally compact, prime and strict.

PROOF. (1) ⇒ (2) The set $\Lambda(s,t)$ is conditionally compact and hence bounded in $\mathcal{L}(X)$. Thus (s,t) has equibounded iterates. Since $s^\wedge(M) = t^\wedge(M) = 1$, Lemma 3.2 implies that

$$p(1,1) \leq \|p(s,t)\| \leq Mp(1,1)$$

for all $p(s,t) \in \mathcal{P}(s,t)$. By Lemma 3.1 the set

$$\Gamma = \{p(s,t) \in \mathcal{P}(s,t) : 0 < p(s,t) \leq 1\}$$

is conditionally compact, since $\Lambda(s,t)$ is conditionally compact. Thus the semi-algebra $\mathcal{A}(s,t)$ is locally compact.

(2) ⇒ (1) Since $\mathcal{A}(s,t)$ is locally compact and $\Lambda(s,t)$ is uniformly bounded, it is clear that $\Lambda(s,t)$ is conditionally compact.

(2) ⇒ (3) Since $\Lambda(s,t)$ is uniformly bounded, there exists $K > 0$ such that $\|s^i t^j\| \leq K$ for $i + j = 1, 2, \cdots$. By Lemma 3.4 it follows that for each $a \in \mathcal{A}(s,t)$ there exists $\alpha \in \mathbf{R}^+$ such that $a^\wedge(M) = \alpha$ and $\alpha \geq K^{-1}\|a\|$.

Let us now prove that $\mathcal{A}(s,t)$ is prime and strict. Let $a, b \in \mathcal{A}(s,t)$. Then there exist $\alpha, \beta \in \mathbf{R}^+$ such that $a^\wedge(M) = \alpha$, $b^\wedge(M) = \beta$, $\alpha \geq K^{-1}\|a\|$ and $\beta \geq K^{-1}\|b\|$. Thus $(a+b)^\wedge(M) = \alpha + \beta$ and $(ab)^\wedge(M) = \alpha\beta$, whence $a + b \neq 0$ unless $a = b = 0$, and $ab \neq 0$ unless either $a = 0$ or $b = 0$. Thus, $\mathcal{A}(s,t)$ is prime and strict.

(3) $\Rightarrow$ (2) There exists a non-zero projection p such that $sp = r(s)p = p$ and $tp = r(t)p = p$. By Lemma 3.5 there exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$. Because of Corollary 3.3, $r(a) > 0$ for each non-zero $a \in \mathcal{A}(s,t)$. Since the spectral radius is a continuous function on the non-empty compact set $\mathcal{A}_1(s,t) = \{a \in \mathcal{A}(s,t) : \|a\| = 1\}$, there exists an element $a_1 \in \mathcal{A}_1(s,t)$ such that

$$r(a) \geq r(a_1) = \alpha > 0, \quad a \in \mathcal{A}_1(s,t).$$

In particular,

$$r\left(\frac{s^i t^j}{\|s^i t^j\|}\right) \geq \alpha > 0, \quad i + j = 1, 2, \cdots.$$

Thus we have

$$\|s^i t^j\| \leq \frac{1}{\alpha} r(s^i t^j) \leq \frac{1}{\alpha} r(s)^i r(t)^j \leq \frac{1}{\alpha}, \quad i + j = 1, 2, \cdots.$$

Hence $\Lambda(s,t)$ is uniformly bounded. ∎

The converse of Lemma 3.5 requires the assumption that $\Lambda(s,t)$ be conditionally compact, and is not true in general. We restate this fact in the following theorem. In the next section we will provide a counterexample illustrating this assertion.

THEOREM 3.7. *Let s and t be commuting operators in $\mathcal{L}(X)$ such that $r(s) = r(t) = 1$ and $1 \in \sigma(s) \cap \sigma(t)$. Also, let the set $\Lambda(s,t)$ be conditionally compact. Then the following statements are equivalent:*

(1) There exists an element $x \in X$ such that $sx = tx = x$.

(2) There exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$.

PROOF. (1) $\Rightarrow$ (2) This is clear from Lemma 3.5.

(2) $\Rightarrow$ (1) According to Theorem 2.1, $\mathcal{A}(s,t)$ is locally compact. Then Theorem 3.6 implies (1). ∎

REMARK. Theorems 3.6 and 3.7 hold for semi-algebras generated by a finite number of commuting operators if one imposes the same kind of conditions on the operators as in Theorem 3.6.

4. EXAMPLES.

In this section we give some examples of semi-algebras generated by the operators s and t in $\mathcal{L}(X)$. The first example is an application of Theorem 3.6 to spectral operators. The others provide examples of semi-algebras that fail to be either locally compact, prime or strict when we drop either the assumption that s and t have a common eigenvector at the eigenvalue 1 or that there exists an element $M \in \mathcal{M}$ such that $s\hat{}(M) = t\hat{}(M) = 1$. We note that Example 5 is a counterexample to the converse of Lemma 3.5.

EXAMPLE 1. Let $t_1, \cdots, t_m$ be a finite set of commuting scalar-type spectral operators in $\mathcal{L}(X)$ (see [3]). Suppose all of these operators have spectral radius 1 and have 1 in their spectrum. Then

$$t_k{}^i = \int_{\sigma(t_k)} \lambda^i dE_k(\lambda); \; k = 1, 2, \cdots, m; \; i \in \mathbf{N},$$

so that $\|t_1^{i_1} \cdots t_m^{i_m}\| \leq v(E_1) \cdots v(E_m)$, where $v(E_k)$ is the total variation of the spectral measure E_k, $k = 1, 2, \cdots, m$. Hence, $(t_1, \cdots, t_m)$ has equibounded iterates. Then, if there exists a non-zero $x \in X$ such that $t_1 x = \cdots = t_m x = x$, $\mathcal{A}(t_1, \cdots, t_m)$ is locally compact if and only if $t_1, \cdots, t_m$ are Radon-Nikolskii operators. Indeed, if $\mathcal{A}(t_1, \cdots, t_m)$ is locally compact, then each $\mathcal{A}(t_k)$ is locally compact and hence, using Theorem 5 of [5], each t_k is a Radon-Nikolskii operator. The converse statement is immediate from Theorem 2.2.

EXAMPLE 2. Let $X = \ell_1$. Define $s, t \in \mathcal{L}(X)$ by

$$s(a, b, c_1, c_2, \cdots) = (a, -b, 0, 0, \cdots),$$

$$t(a,b,c_1,c_2,\cdots)=(-a,b,\frac{1}{2}c_2,\frac{1}{3}c_3,\cdots).$$

Then $st=ts$ and $\|s\|=\|t\|=1$, so that $\Lambda(s,t)$ is bounded. Since $\lim_{n\to\infty}\|(s+t)^n\|=0$, $|\mu(s+t)|<1$ for every multiplicative linear functional μ on $\mathcal{B}(s,t)$. Hence there does not exist an element $M\in\mathcal{M}$ such that $s\hat{}(M)=t\hat{}(M)=1$.

Let us prove that $\mathcal{A}(s+t)$ is not locally compact. We have

$$\|(s+t)^k\|=\frac{1}{(k+1)!}.$$

Now let $u_k=(k+1)!(s+t)^k$; then $\|u_k\|=1$ and $\lim_{n\to\infty}\|u_k x\|=0$ for all $x\in X$. But then $\{u_k\}_{k=1}^{\infty}$ cannot have a uniformly convergent subsequence, and hence $\mathcal{A}(s+t)$, and therefore $\mathcal{A}(s,t)$, is not locally compact.

EXAMPLE 3. Let $X=\mathbf{C}^3$. Define $s,t\in\mathcal{L}(X)$ by

$$s(a,b,c)=(-a,-b,c),\quad t(a,b,c)=(a,-b,-c).$$

Then $st=ts$, $\|s\|=\|t\|=1$ and $s^2=t^2=I$. Then $\Lambda(s,t)$ is bounded and $\mathcal{A}(s,t)=\{\alpha I+\beta s+\gamma t+\delta st:\alpha,\beta,\gamma,\delta\geq 0\}$ is a subset of a finite-dimensional space and hence locally compact. However, $(I+s+t+st)=0$, and hence $\mathcal{A}(s,t)$ is not strict.

EXAMPLE 4. Let $X=\ell_1$. Define $s,t\in\mathcal{L}(X)$ by

$$s(a_1,a_2,\cdots)=(a_1,0,\frac{1}{2}a_3,\frac{1}{2}a_4,\cdots),$$

$$t(a_1,a_2,\cdots)=(0,a_1,0,0,\cdots).$$

Then $st=ts=0$, so that $\mathcal{A}(s,t)$ is not prime. Also, since $\|s\|=\|t\|=1$, $\Lambda(s,t)$ is bounded. Nevertheless, $\mathcal{A}(s,t)$ is locally compact. Indeed, since $st=0$,

$$\begin{aligned}\mathcal{A}(s,t)&=\left\{\beta_1 t+\beta_2 p+\sum_{i=1}^{\infty}a_i s^i:\beta_1,\beta_2,\alpha_i\geq 0,i\in\mathbf{N},\ and\ \sum_{i=1}^{\infty}a_i<\infty\right\}\\&=\mathcal{A}(t)\oplus\mathcal{A}(s),\end{aligned}$$

where $p=\lim_{n\to\infty}s^n$ is given by $p(a_1,a_2,\cdots)=(a_1,0,0,\cdots)$. Since both $\mathcal{A}(s)$ and $\mathcal{A}(t)$ are locally compact, $\mathcal{A}(s,t)$ is locally compact.

EXAMPLE 5. Let $X=C[0,1]$. Put $y(x)=x$ for all $x\in[0,1]$. Define $s,t\in\mathcal{L}(X)$ by

$$sf=yf,\qquad tf=y^{\sqrt{2}}f,\quad f\in C[0,1].$$

Then $\mathcal{B}(I,s,t)$ is isomorphic to $C[0,1]$, so that its maximal ideal space coincides with $[0,1]$. Thus

$$s\hat{}(1)=y(1)=1,\quad t\hat{}(1)=y^{\sqrt{2}}(1)=1.$$

However, the operators s and t do not have 1 as an eigenvalue.

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