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OSCILLATION IN NEUTRAL EQUATIONS WITH AN "INTEGRALLY SMALL" COEFFICIENT

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ABSTRACT. Consider the neutral delay differential equation

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + Q(t)x(t-\delta) = 0, t \geqslant t_0 \qquad (*)$$

Where $P,Q \in C([t_o,\infty],R^+)$, $\tau \in (o,\infty)$ and $\delta \in R^+$. We obtain several sufficient conditions for the oscillation of all solutions of Eq. $(*)$ without the restriction

$$\int_{t_0}^{\infty} Q(s)ds = \infty.$$

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1. INTRODUCTION.

In this paper we consider the first order neutral delay differential equation

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + Q(t)x(t-\sigma) = 0, t \geqslant t_0 \qquad (1)$$

where

$$\tau \in (0,\infty), \sigma \in R^+ \textit{ and } P,Q \in C([t_0,\infty),R^+) \qquad (2)$$

Our aim in this paper is to establish some sufficient conditions for the oscillation of all solutions of Eq. (1) which does not require that

$$\int_{t_0}^{\infty} Q(s)ds = \infty \qquad (3)$$

The oscillatory behavior of Eq. (1) has been investigated by many authors,see for example[1—2] and [4—7]. For a recent survey,we can see [3]. The most of the papers in the literature,however,concern the equation (1) under the hypothesis(3).

Moreover, (3) is also a sufficient condition for the oscillation of Eq. (1) with $P(t)\equiv 1$. Therefore, Chuanxi and Ladas put forward the following open problem in [1].

OPEN PROBLEM A[1]. Is condition (3) a necessary condition for the oscillation of all solutions of Eq. (1) with $P(t)\equiv 1$?

Recently, Yu, Wang and Chuanxi [6] answered the above problem and proved the following result in [6].

THEOREM B [6]. Assume that (2) holds with $P(t)\equiv 1$. Suppose also

$$\int_{t_0}^{\infty} sQ(s)\left[\int_{s}^{\infty} Q(t)dt\right]ds = \infty. \tag{4}$$

Then every solution of Eq. (1) oscillates.

Clearly, condition (4) is better than (3). Therefore, the answer to open problem A is negative.

In [8], Zhang and Yu prove the following theorem

THEOREM C [8]. Assume that (2) holds with $P(t)\equiv 1$. Then every bounded solution of Eq. (1) oscillates if and only if

$$\int_{t_0}^{\infty} sQ(s)ds = \infty \tag{5}$$

In this paper we will establish several further oscillation results for Eq. (1) when condition (3) does not satisfy, that is, when $Q(t)$ is "integrally small". This is done by using Lemmas 1 and 2 given in section 2. These lemmas are interesting in their own right.

In the sequel, for the sake of convenience, we define

$$H_0(t) = \int_{t}^{\infty} Q(s)ds$$

Let $\rho = max\{\tau,\delta\}$. By a solution of Eq. (1) we mean a function $x\in C([t_1-\rho,\infty],R)$, for some $t_1\geqslant t_0$, such that $x(t)-P(t)x(t-\tau)$ is continuously differentiable on $[t_1,\infty)$ and satisfies Eq. (1) for $t\geqslant t_1$.

Assume that (2) holds, $t_1\geqslant t_0$ and let $\varphi\in C([t_1-\rho,t_1],R)$ be a given initial function. Then we can easily see by the method of steps that Eq. (1) has a unique solution $x\in C([t_1-\rho,\infty),R)$ such that

$$x(t) = \varphi(t) \ for\ t_1-\rho\leqslant t\leqslant t_1.$$

As is customary, a solution of Eq. (1) is said to oscillate if it has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

2. TWO IMPORTANT LEMMAS

At first, we assume

(Y_1) There exists a $t^*\geqslant t_0$ such that

$$P(t^*+i\tau)\leqslant 1 \quad for\ i = 0,1,2,\cdots \tag{7}$$

and

(Y_2) There exists a nonnegative integer k such that the functions

$$H_m(t) = \int_{t}^{\infty} sQ(s)H_{m-1}(s)ds,\ m = 1,2,\cdots,k \tag{8}$$

exist and

$$\int_{t_0}^{\infty} sQ(s)H_k(s)ds = \infty \tag{9}$$

The main results in this section are Lemmas 1 and 2 which enable us to establish some new type of oscillation criteria for Eq. (1).

LEMMA 1. Assume that (2) and (Y_1) hold. Suppose also that $Q(t)$ is not identically zero on any half infinite interval $[T,\infty)$, $T\geqslant t_0$. Let $x(t)$ be an eventually positive solution of the differential inequality

$$\frac{d}{dt}[x(t) - P(t)x(t-\tau)] + Q(t)x(t-\sigma) \leqslant 0 \tag{10}$$

and set

$$y(t) = x(t) - P(t)x(t-\tau). \tag{11}$$

Then we have eventually

$$y(t) > 0. \tag{12}$$

PROOF. Let $t_1\geqslant t_0$ be such that

$$x(t-\rho) > 0 \; for \; t \geqslant t_1$$

where $\rho=max\{\tau,\delta\}$. Then by (10)and (11) we have

$$y'(t) \leqslant - Q(t)x(t-\delta) \leqslant 0 \; for \; t \geqslant t_1,$$

which implies that $y(t)$ is nonincreasing on $[t_1,\infty)$. Hence, if (12) does not hold, then eventually

$$y(t) < 0.$$

Therefore, there exist $t_2>t_1$ and $\alpha>0$ such that

$$y(t) \leqslant -\alpha \quad for \; t \geqslant t_2$$

That is,

$$x(t) \leqslant -\alpha + P(t)x(t-\tau) \quad for \; t \geqslant t_2 \tag{13}$$

Now choose a positive integer n^* such that $t^*+n^*\tau\geqslant t_2$. Then by (Y_1) and (13) we get

$$x(t^* + n^*\tau + j\tau) \leqslant - j\alpha + x(t^* + n^*\tau)$$
$$\rightarrow -\infty \; as \; j \rightarrow \infty.$$

which contradicts the positivity of $x(t)$. And this proof is complete.

REMARK 1. Lemma 1 is an improvement of Lemma 1 in [2] since the condition $Q(t)\geqslant q>0$ for $t\geqslant t_0$ is relaxed.

LEMMA 2. Assume that (2) and (Y_2) hold and that

$$P(t) \geqslant 1 \quad for \; t \geqslant t_0 \tag{14}$$

Let $x(t)$ be an eventually positive solution of inequality (10) and let $y(t)$ be defined by(11). Then eventually

$$y(t) < 0. \tag{15}$$

PROOF. From(1)we have eventually

$$y'(t) \leqslant - Q(t)x(t-\delta) \leqslant 0 \tag{16}$$

Therefore, if (15) does not hold, then eventually

$$y(t) > 0 \tag{17}$$

In this case, we note that $x(t)>x(t-\tau)$ eventually. This implies that there exist $M>0$ and $t_1 \geqslant t_0$ such that $x(t-\rho) \geqslant M$ for $t \geqslant t_1$, where $\rho = max\{\tau, \delta\}$. Then by (16), it follows that

$$y'(t) \leqslant - MQ(t), for\ t \geqslant t_1$$

and so

$$y(t) \geqslant M\int_t^\infty Q(s)ds = MH_0(t), for\ t \geqslant t_1,$$

which, together with (11) and (14), yields

$$x(t) \geqslant x(t-\tau) + MH_0(t)\ for\ t \geqslant t_1 \tag{18}$$

Let $m(t)$ denote the greatest integer part of $\frac{t-t_1}{\tau}$. Then we have

$$\begin{aligned} x(t) &\geqslant M[H_0(t) + H_0(t-\tau) + \cdots + H_0(t-(m(t)-1)\tau)] \\ &\quad + x(t-m(t)\tau) \\ &> M[H_0(t) + H_0(t-\tau) + \cdots + H_0(t-(m(t)-1)\tau)], \\ &\quad for\ t \geqslant t_1 + \tau. \end{aligned}$$

Using the fact that $H_0(t)$ is decreasing, we get

$$x(t) > Mm(t)H_0(t), for\ t \geqslant t_1 + \tau$$

Substituting this into (16), we obtain

$$\begin{aligned} y'(t) &\leqslant - MQ(t)m(t-\delta)H_0(t-\delta) \\ &\leqslant - Mm(t-\delta)Q(t)H_0(t), for\ t \geqslant t_1 + \tau + \delta \end{aligned} \tag{19}$$

By noting that $t/m(t-\delta) \to \tau$ as $t \to \infty$, we see that

$$\frac{Mm(t-\delta)Q(t)H_0(t)}{tQ(t)H_0(t)} \to \frac{M}{\tau}\ as\ t \to \infty.$$

It follows that there exists $t_2 > t_1 + \tau + \delta$ such that

$$Mm(t-\delta)Q(t)H_0(t) \geqslant \frac{M}{2\tau} tQ(t)H_0(t), for\ t \geqslant t_2$$

and so

$$y'(t) \leqslant -\frac{M}{2\tau} tQ(t)H_0(t), for\ t \geqslant t_2. \tag{20}$$

If $k=0$, then a contradiction can be easily derived by directlly integrating (20). Therefore, it suffices to show that another contradiction is also derived in the case $k \neq 0$. Indeed, by directly integrating (20) from $t \geqslant t_2$ to ∞, we find

$$y(t) \geqslant \frac{M}{2\tau} H_1(t)$$

which, together with (11) and (14), yields

$$x(t) \geqslant x(t-\tau) + \frac{M}{2\tau} H_1(t), for\ t \geqslant t_2 \tag{21}$$

Now, if we let $m(t)$ denote the greatest integer part of $\frac{t-t_2}{\tau}$, then

$$\begin{aligned} x(t) &\geqslant \frac{M}{2\tau}[H_1(t) + H_1(t-\tau) + \cdots + H_1(t-(m(t)-1)\tau)] \\ &+ x(t-m(t)\tau) \\ &> \frac{M}{2\tau}[H_1(t) + H_1(t-\tau) + \cdots + H_1(t-(m(t)-1)\tau)], \\ &for\ t \geqslant t_2 + \tau \end{aligned}$$

Also as $H_1(t)$ is nonincreasing, it follows that

$$x(t) \geqslant \frac{M}{2\tau} m(t) H_1(t), for\ t \geqslant t_2 + \tau$$

By a diret substitution, we get

$$\begin{aligned} y'(t) &\leqslant -\frac{M}{2\tau} m(t-\delta)Q(t)H_1(t-\delta) \\ &\leqslant -\frac{M}{2\tau} m(t-\delta)Q(t)H_1(t), for\ t \geqslant t_2 + \tau + \delta \end{aligned}$$

From the fact that $m(t-\delta)/t \to \frac{1}{\tau}$ as $t \to \infty$, we see that there exists $t_3 \geqslant t_2 + \tau + \delta$ such that

$$m(t-\delta) \geqslant \frac{t}{2\tau} \quad for\ t \geqslant t_3$$

Hence

$$y'(t) \leqslant -\frac{M}{2^2\tau^2} tQ(t)H_1(t), for\ t \geqslant t_3$$

which implies

$$y(t) \geqslant \frac{M}{4\tau^2} H_2(t), for\ t \geqslant t_3.$$

This, together with (11) and (14), implies

$$x(t) \geqslant x(t-\tau) + \frac{M}{2^2\tau^2} H_2(t), for\ t \geqslant t_3. \tag{22}$$

Thus, by using induction, we can get, for some sufficiently large t_{k+2},

$$y'(t) \leqslant -\frac{M}{2^{k+1}\tau^{k+1}} tQ(t)H_k(t), for\ t \geqslant t_{k+2}$$

which, together with (Y_2), yields

$$y(t) \to -\infty\ as\ t \to \infty,$$

which contradicts the hypothesis that $y(t)$ is eventually positive. Therefore, (15) holds and the proof is complete.

3. MAIN RESULTS

In this section we will apply the lemmas in section 2 to establish several new oscillation criteria for Eq. (1) without (3). The following Theorem 1 is an immediete corollary of Lemmas 1 and 2, which contains theorem 1 in [6] as a special case when $k = 0$.

THEOREM 1. Assume that (2) and (Y_2) hold with $P(t) \equiv 1$. Then every solution of Eq. (1) oscillates.

EXAMPLE 1. Consider the neutral delay differential equation

$$\frac{d}{dt}[x(t) - x(t-\tau)] + \frac{1}{t^{\alpha}} x(t-\delta) = 0 \tag{23}$$

Where $1 < \alpha < 2$. Since

$$\lim_{k\to\infty} \frac{2k+3}{k+2} = 2 > \alpha$$

it follows that there exists a least positive integer k such that

$$\frac{2k+3}{k+2} \geqslant \alpha$$

Since

$$H_m(t) = \frac{1}{(\alpha-1)(2\alpha-3)\cdots((m+1)\alpha-(2m+1))} \cdot \frac{1}{t^{(m+1)\alpha-(2m+1)}},$$
$$m = 0, 1, \cdots, k$$

exist, we find that (Y_2) holds. Therefore, by Theorem 1, every solution of Eq. (23) oscillates. But, condition (4) is not satisfied when $\frac{3}{2} < \alpha < 2$. On the other hand, by Theorem C, we see that Eq. (23) has a bounded nonoscillatory solution if and only if $\alpha > 2$. Now we do not know how to handle the case $\alpha = 2$. Therefore, it remains an open problem to determine the oscillation of all solutions of Eq. (23) with $\alpha = 2$.

THEOREM 2. Assume that (2), (Y_1) and (Y_2) hold. Suppose also that

$$P(t-\delta)Q(t) \geqslant Q(t-\tau) \quad for\ t \geqslant t_0+\rho \tag{24}$$

Then every solution of Eq. (1)oscillates.

PROOF. Otherwise, Eq. (1)has an eventually positive solution $x(t)$. Let $y(t)$ be defined by (11). Then by Lemma 1 we have for some $t_1 \geqslant t_0$,

$$y(t) > 0 \quad for\ t \geqslant t_1$$

By direct substitution one can see that

$$y'(t) = -Q(t)x(t-\delta) \leqslant 0, for\ t \geqslant t_1+\delta$$

which, together with(11)and(24), implies

$$\begin{aligned} y'(t) &= -Q(t)[y(t-\delta)+P(t-\delta)x(t-\delta-\tau)] \\ &\leqslant -Q(t)y(t-\delta)-Q(t-\tau)x(t-\delta-\tau) \\ &= -Q(t)y(t-\delta)+y'(t-\tau), for\ t \geqslant t_1+\delta+\tau \end{aligned}$$

That is,

$$y'(t)-y'(t-\tau)+Q(t)y(t-\delta) \leqslant 0, for\ t \geqslant t_1+\delta+\tau \tag{25}$$

By Lemma 1 and 2, we see that (Y_2) implies that(25)has no eventually positive solutions. This is a contradiction and so the proof is complete.

The following result is an immediate corollary of Theorem 2.

COROLLARY 1. Assume that (2) and (24) hold and that

$$0 \leqslant P(t) \leqslant 1 \quad for\ t \geqslant t_0 \tag{26}$$

Suppose also that $Q(t)$ is not identically zero on any half interval $[T, \infty], T \geqslant t_0$. Then every solution of Eq. (1) oscillates.

This is because (24) and (26) imply

$$Q(t) \geqslant Q(t-\tau), for\ t \geqslant t_0+\rho$$

which, together with the fact that $Q(t)$ is not identically zero eventually, yields

$$\int_{t_0}^{\infty} Q(s)ds = \infty.$$

Thus, (Y_2) holds. Therefore, by Theorem 2, every solution of Eq. (1)oscillates.

THEOREM 3. Assume that (2), (Y_2) and (14) hold. Suppose also that

$$P(t-\delta)Q(t) \leqslant Q(t-\tau), for\ t \geqslant t_0+\tau+\delta \tag{27}$$

Then every solution of Eq. (1)oscillates.

PROOF. Otherwise, Eq. (1) has an eventually positive solution $x(t)$. Now we set $y(t)$ as in(11). Then by Lemma 2 we have for some $t_1 \geqslant t_0$,

$$y(t) < 0, for\ t \geqslant t_1 \tag{28}$$

Clearly, by (11) and (27) we have

$$\begin{aligned} y'(t) &= -Q(t)x(t-\delta) \\ &= -Q(t)[y(t-\delta)+P(t-\delta)x(t-\delta-\tau)] \\ &\geqslant -Q(t)y(t-\delta)-Q(t-\tau)x(t-\delta-\tau) \\ &= -Q(t)y(t-\delta)+y'(t-\tau), for\ t \geqslant t_1+\tau+\delta \end{aligned}$$

which implies that $-y(t)$ is a positive solution of the inequality

$$z'(t)-z'(t-\tau)+Q(t)z(t-\delta) \leqslant 0$$

Thus, by directly using Lemma 1 and 2, we may obtain a contradiction and so the proof of this theorem is complete.

EXAMPLE 2. The neutral delay differential equation

$$\frac{d}{dt}[x(t) - \frac{t+1}{t}x(t-1)] + \frac{1}{\sqrt[3]{t^5}}x(t-1) = 0, t \geqslant 1 \tag{29}$$

satisfies the conditions of Theorem 3. Therefore every solution of Eq. (29) oscillates.

The following theorem can be easily proved by directly appling Lemma 1 and 2.

THEOREM 4. Assume that(2), $(Y_1)-(Y_2)$ and(14) hold. Then every solution of Eq. (1) oscillates.

EXAMPLE 3. The neutral delay differential equation

$$\frac{d}{dt}[x(t) - (2+\sin t)x(t-2\pi)] + \frac{1}{t^\alpha}x(t) = 0, t \geqslant 1$$

satisfies the all hypotheses of Theorem 4 when $\alpha<2$. Therefore, every solution of this equation oscillates when $\alpha<2$.

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