RESEARCH NOTES

A NOTE ON DUAL INTEGRAL EQUATIONS INVOLVING ASSOCIATED LEGENDRE FUNCTION

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ABSTRACT. This note is concerned with a formal method to obtain a closed form solution of certain dual integral equations involving the associated Legendre function $P^{\mu}_{-1/2+i\tau}(\cosh\alpha)$ as kernel.

KEY WORDS AND PHRASES: Dual integral equations and associated Legendre function as kernal.
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1. INTRODUCTION.

Dual integral equations involving the associated Legendre function $P^{m}_{-1/2+\tau}(\cosh\alpha)$ $(m = 0, 1, 2, \ldots)$ were considered by Rukhovets and Ufliand [1] who expressed the solution in terms of one unknown function which satisfies a Fredholm integral equation of second kind. Later Pathak [2] considered several dual integral equations involving $P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)$ where $-\frac{1}{2} < Re\,\mu < \frac{1}{2}$. He exploited the results of some integrals involving $P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)$ to handle these dual integral equations and obtained closed form solution in some cases and reduced them to the solution of a Fredholm integral equation of second kind in other cases. For $\mu = 0$ the corresponding integral equations were mostly considered earlier by Babloian [3].

In the present note we have considered a closed form solution of the following dual integral equations

$$\int_0^\infty \tau^{-1}A(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\tanh c\tau\, d\tau = f(\alpha), \quad 0 < \alpha < a, \tag{1.1}$$

$$\int_0^\infty A(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)d\tau = g(\alpha), \quad \alpha > a, \tag{1.2}$$

with $Re\,\mu < \frac{1}{2}$. We first find a closed form solution of the pair of dual integral equations consisting of (1.1) and (1.2) with $g(\alpha) = 0$, and then the solution of the pair consisting of (1.1) with $f(\alpha) = 0$ and (1.2). The desired solution of the dual integral equations (1.1) and (1.2) is then obtained by combining solutions of these two pairs. In both cases results of two integrals involving $P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)$ have been utilized to handle them. These are given by (cf. [2])

$$(2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\,\frac{d}{dx}\int_0^x \frac{\sinh^{1-\mu}\alpha P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)}{(\cosh x-\cosh\alpha)^{1/2-\mu}}\,d\alpha = \cos\tau x, \tag{1.3}$$

and

$$\left(\frac{2}{\pi}\right)^{1/2}\Gamma\left(\frac{1}{2}-\mu\right)\sinh^{-\mu}\alpha\int_0^\infty P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\cos\tau t\, d\tau = \begin{cases}(\cosh\alpha-\cosh t)^{-\mu-\frac{1}{2}}, & t<\alpha\\ 0 & , t>\alpha\end{cases} \tag{1.4}$$

where $Re\,\mu < \frac{1}{2}$.

The case $\mu = 0$, $g(\alpha) = 0$ was considered earlier by Dhaliwal and Singh [4] who obtained a closed form solution of the corresponding dual integral equations. The method used in this note is formal and may be regarded as a generalization of the method given in [4]. No attempt has been made to find precise conditions under which the solution obtained here is valid.

2. SOLUTION WHEN $g(\alpha) = 0$

In this section we find a closed form solution of the following dual integral equations

$$\int_0^\infty \tau^{-1} B(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\tanh c\tau\, d\tau = f(\alpha), \quad 0<\alpha<a, \tag{2.1}$$

$$\int_0^\infty B(\tau) P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\, d\tau = 0, \quad \alpha > a, \tag{2.2}$$

with $Re\mu < \frac{1}{2}$.

Multiplying (2.1) by $\pi^{-1}\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\sinh^{1-\mu}\alpha(\cosh x - \cosh\alpha)^{\mu-\frac{1}{2}}$, integrating the result with respect to α from 0 to x where $x<a$, and then differentiating both sides with respect to x we find after utilizing the result (1.3) that

$$\int_0^\infty \tau^{-1} B(\tau)\tanh c\tau \cos\tau x\, d\tau = F(x), \quad 0<x<a, \tag{2.3}$$

where

$$F(x) = (2\pi)^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\mu\right)\cos\mu\pi\frac{d}{dx}\int_0^x \frac{\sinh^{1-\mu}\alpha f(\alpha)}{(\cosh x - \cosh\alpha)^{1-\mu}}\, d\alpha. \tag{2.4}$$

Let us assume that

$$B(\tau) = \int_0^a \phi(t)\cos\tau t\, dt \tag{2.5}$$

where $\phi(t)$ is to be found out from (2.3) and (2.2). If we substitute (2.5) in (2.3) and interchange the order of integration (assuming this to be valid) we find

$$\frac{1}{2}\int_0^a \phi(t) ln \left| \frac{\cosh\frac{\pi x}{2c} + \cosh\frac{\pi t}{2c}}{\cosh\frac{\pi x}{2c} - \cosh\frac{\pi t}{2c}} \right| dt = F(x), \quad 0<x<a,$$

which, after differentiation with respect to x produces the singular integral equation for $\phi(t)$ given by

$$\int_0^a \frac{\phi(t)\cosh\frac{\pi t}{2c}}{\cosh\frac{\pi t}{c} - \cosh\frac{\pi x}{c}}\, dt = \frac{c}{\pi}\frac{F'(x)}{\sinh\frac{\pi x}{c}}, \quad 0<x<a, \tag{2.6}$$

where the integral is in the sense of the Cauchy principal value. If we substitute

$$\left.\begin{aligned} &\xi = \cosh\frac{\pi t}{2c}, \quad \eta = \cosh\frac{\pi x}{c}, \quad k = \cosh\frac{\pi a}{c}, \\ &\Phi(\xi) = \phi(t)/\sinh\frac{\pi t}{2c}, \quad L(\eta) = -\frac{2}{\pi}\frac{F'(x)}{\sinh\frac{\pi x}{c}}, \end{aligned}\right\} \tag{2.7}$$

then (2.6) reduces to the familiar form

$$\int_1^k \frac{\Phi(\xi)}{\eta-\xi}\, d\xi = \pi L(\eta), \quad 1<\eta<k.$$

Its solution can be written in several different forms (cf. Cooke [5]). One such form is

$$\Phi(\xi) = \frac{1}{\pi}\left\{\frac{\xi-1}{k-\xi}\right\}^{\frac{1}{2}}\int_1^k \left\{\frac{k-\eta}{\eta-\eta}\right\}^{\frac{1}{2}}\frac{L(\eta)}{\eta-\xi}\, d\eta + \frac{D_1}{\{(\xi-1)(k-\xi)\}^{1/2}}$$

where D_1 is an arbitrary constant. The back substitution from (2.7) produces the desired solution in the form

$$\phi(t) = -\frac{4\sqrt{2}}{\eta c}\,\chi(t) + \frac{D_1}{\sqrt{2}}\left(\cosh\frac{\pi a}{c} - \cosh\frac{\pi t}{c}\right)^{-1/2} \tag{2.8}$$

where

$$\chi(t) = \frac{\sinh^2\frac{\pi t}{c}}{\left(\cosh\frac{\pi a}{c} - \cosh\frac{\pi t}{c}\right)^{1/2}} \int_0^a \left\{\frac{\cosh\frac{\pi a}{c} - \cosh\frac{\pi x}{c}}{\cosh\frac{\pi x}{c} - 1}\right\}^{\frac{1}{2}} \frac{F'(x)\cosh\frac{\pi x}{2c}}{\cosh\frac{\pi x}{c} - \cosh\frac{\pi t}{c}}\,dx\,. \tag{2.9}$$

To find the arbitrary constant D_1 in (2.8) we use (2.5) in (2.2) and utilizing the result (1.4) we find that

$$\int_0^a \frac{\phi(t)}{(\cosh\alpha - \cosh t)^{\mu+1/2}}\,dt = 0\,, \quad \alpha > a\,. \tag{2.10}$$

If we multiply both sides of (2.10) by $\sinh\frac{\alpha}{2}\left(\cosh\frac{\alpha}{2}\right)^{2\mu-2}$ and integrate the result with respect to α from a to ∞, we obtain

$$2^{1/2-\mu}\int_0^a \frac{\phi(t)}{\cosh^2\frac{t}{2}}\,dt = \cosh^{2\mu-1}\frac{a}{2}\int_0^a \frac{(\cosh a - \cosh t)^{-\mu+1/2}}{\cosh^2\frac{t}{2}}\,\phi(t)dt \tag{2.11}$$

If we utilize (2.8) in (2.11) we obtain a linear equation for D_1 from which it is found that

$$D_1 = \frac{J_1 - J_2}{I_1 - I_2} \tag{2.12}$$

where

$$I_1 = 2^{1/2-\mu}\int_0^a \frac{dt}{\cosh^2\frac{t}{2}\left(\cosh\frac{\pi a}{c} - \cosh\frac{\pi t}{c}\right)^{1/2}}\,,$$

$$I_2 = \cosh^{2\mu-1}\frac{a}{2}\int_0^a \frac{(\cosh a - \cosh t)^{-\mu+1/2}}{\cosh^2\frac{t}{2}\left(\cosh\frac{\pi a}{c} - \cosh\frac{\pi t}{c}\right)^{1/2}}\,dt \tag{2.13}$$

and

$$J_1 = \frac{8}{\pi c}\int_0^a \frac{\chi(t)}{\cosh^2\frac{t}{2}}\,dt\,,$$

$$J_2 = \frac{8}{\pi c}\cosh^{2\mu-1}\frac{a}{2}\int_0^a \frac{(\cosh a - \cosh t)^{-\mu+1/2}}{\cosh^2\frac{t}{2}}\,\chi(t)dt\,. \tag{2.14}$$

We note that for the special case of $\mu = 0$, the result of [3] is recovered. Since $\phi(t)$ is now known completely, a closed form solution of the dual integral equations (2.1) and (2.2) is obtained by using this $\phi(t)$ in (2.5).

3. SOLUTION WHEN $f(\alpha) = 0$

In this section we find a closed form solution of the dual integral equations

$$\int_0^\infty \tau^{-1}C(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)\tanh c\tau d\tau = 0\,, \quad 0 < \alpha < a\,, \tag{3.1}$$

$$\int_0^\infty C(\tau)P^{\mu}_{-\frac{1}{2}+i\tau}(\cosh\alpha)d\tau = 0\,, \quad \alpha > a\,, \tag{3.2}$$

with $Re\mu < \frac{1}{2}$. By using a procedure similar to that of section 2, (3.1) reduces to

If we assume as before

$$C(\tau) = \int_0^a \psi(t)\cos\tau t\,dt\,, \tag{3.4}$$

then (3.2) gives after utilizing the results (1.4) that

$$\int_0^a \frac{\psi(t)}{(\cosh\alpha - \cosh t)^{\mu+1/2}}\,dt = G(\alpha)\,, \quad \alpha > a \tag{3.5}$$

where

$$G(\alpha) = \left(\frac{2}{\pi}\right)^{1/2}\Gamma\left(\frac{1}{2}-\mu\right)\sinh^{-\mu}\alpha\, g(\alpha)\,. \tag{3.6}$$

Using (3.4) in (3.3) we find that $\chi(t)$ satisfies the homogeneous singular integral equation

$$\int_0^a \frac{\psi(t)\cosh\frac{\pi t}{2c}}{\cosh\frac{\pi t}{c} - \cosh\frac{\pi x}{c}}\,dt = 0\,, \quad 0 < x < a\,. \tag{3.7}$$

Its solution is given by

$$\psi(t) = \frac{D_2}{\sqrt{2}}\left(\cosh\frac{\pi a}{c} - \cosh\frac{\pi t}{c}\right)^{-1/2} \tag{3.8}$$

where D_2 is an arbitrary constant which has to be found from (3.5). For this purpose we multiply both sides of (3.5) by $\sinh\frac{\alpha}{2}\left(\cosh\frac{\alpha}{2}\right)^{2\mu-2}$ and integrate with respect to α from a to ∞ to obtain

$$2^{1/2-\mu}\int_0^a \frac{\psi(t)}{\cosh^2\frac{t}{2}}\,dt - \cosh^{2\mu-1}\frac{a}{2}\int_0^a \frac{(\cosh a - \cosh t)^{-\mu+\frac{1}{2}}}{\cosh^2\frac{t}{2}}\,\psi(t)dt$$

$$= (1-2\mu)\int_a^\infty \sinh\frac{\alpha}{2}\left(\cosh\frac{\alpha}{2}\right)^{2\mu-2} G(\alpha)d\alpha = I\,, \quad \text{say}\,. \tag{3.9}$$

Substituting (3.8) in (3.9) we obtain a linear equation for D_2 form which we find

$$D_2 = \frac{I}{I_1 - I_2} \tag{3.10}$$

where I_1 and I_2 are given in (2.13).

Thus $\psi(t)$ is now completely determined so that a closed form solution of the dual integral equations (3.1) and (3.2) is obtained by using this $\psi(t)$ in (3.4). Solution of the original dual integral equations can now be obtained in a closed form by adding the solutions $B(\tau)$ and $C(\tau)$ of the pairs (2.1), (2.2) and (3.1) and (3.2) respectively.

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